SOME ASPECTS OF ANOMALOUS TRANSPORT DUE TO STOCHASTIC MAGNETIC FIELDS

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ABSTRACT

A variety of diagnostics indicate the presence, in a hot magnetized plasma, of a wide spectrum of fluctuations, $\delta\tilde{n}$, $\delta\tilde{\mathbf{E}}$, $\delta\tilde{\mathbf{B}}$, $\delta\tilde{T}$, which are probably due to micro-instabilities. The theories of anomalous transport try to explain the observed transport in terms of the measured fluctuations. We consider some recent applications of stochastic methods to the study of plasmas in a turbulent state.

I. PARTICLES IN TURBULENT MAGNETIC FIELD

We shall consider different approaches to the problem of transport in a turbulent magnetic fields in toroidal magnetic fields. This problem is of fundamental importance in fusion since flow of particles and heat are very rapid along the magnetic field lines which may be open, due to radial magnetic field perturbations $\delta\tilde{B}_r$. Stochasticity of the field lines can thus yield to an important radial transport and therefore to an important loss of energy.

For the discussion of the methods it is sufficient to consider a magnetic field of the form

$$\mathbf{B} = B_0[\mathbf{e}_z + b_y(z)\,\mathbf{e}_y + b_x(z)\,\mathbf{e}_x]\,, \tag{1}$$

satisfying the constraint $\nabla\cdot\mathbf{B}$. The full generality of the problem is not considered since we assume $b_x(z)$ and $b_y(z)$ constant in the plane perpendicular to $\mathbf{B}$. The random fluctuating field $\mathbf{b}$ is a Gaussian stochastic process characterized by a single spatial scale (a parallel correlation length) $\lambda_\parallel$. Its (two point) Eulerian correlation is given by $(m,n=x,y)$

$$< b_m(z)\,b_n(z') >= \beta^2 \exp\left(-\frac{(z-z')^2}{2\lambda_\parallel^2}\right)\delta_{mn}\,. \tag{2}$$

In the fourier space, with $b_m(z)=\int dk\,e^{ikz}\,b_m(k)$, the correlation function becomes

$$< b_m(k)\,b_n(k') >= \mathcal{B}(k)\,\delta(k+k')\,\delta_{mn}\,, \tag{3}$$

where the spectral density of the magnetic field fluctuations is

$$\mathcal{B} = \frac{\lambda_\parallel\,\beta^2}{\sqrt{2\pi}}\exp\bigl(-\frac{1}{2}\lambda_\parallel^2 k^2\bigr)\,. \tag{4}$$

We now consider a test particle in this stochastic magnetic field. If the averaged magnetic field is sufficiently strong, the position of the particle is assimilated to the position of its guiding centre and its motion is mainly parallel to the perturbed magnetic field with suddent interruptions due to collisions with the particles of the background plasma. This later effect is modelled either by a random variation of the velocity field or by a random variation of the acceleration. Various models, usually gaussian, can be used for the velocity: white noise or colored noise [1]. The analysis can also be extended to a two component background with different thermal velocities (and non gaussian acceleration). The Langevin formalism gives the following equations of motion:

$$\begin{aligned}\frac{d\mathbf{x}_\perp(t)}{dt} &= \mathbf{b}[\mathbf{x}_\perp, z]\,v_z(t)\\ \frac{dz(t)}{dt} &= v_z(t)\\ \frac{dv_z(t)}{dt} &= -\nu_z\,v_z + \alpha_z(t)\end{aligned} \tag{5}$$

The perpendicular collisions are neglected here to stress the importance of the magnetic fluctuations. From here on, the subscript z will not be written down any more. The last equation in (5) may be used to modelize the parallel velocity by a colored noise process starting from a white noise process for the parallel acceleration α_n:

$$< \alpha_n(t) >= 0,\ < \alpha_n(t)\ \alpha_n(t+\tau) >= A\ \delta(\tau)\,. \tag{6}$$

The formal solution of the last equation of (5) is

$$v = e^{-\nu t}\,\bar{v}_0 + \int_0^t d\tau\,\alpha(\tau)\,e^{-\nu(t-\tau)}\,, \tag{7}$$

where $\bar{v}_0$ is the initial stochastic velocity taken in a maxwellian distribution:

$$\phi(\bar{v}_0) = \frac{1}{(\sqrt{\pi}\,\bar{v}_T)^3}\,e^{-\bar{v}_0^2/\bar{v}_T^2} \tag{8}$$

where $\bar{v}_T$ is the thermal velocity. Recalling (7), and performing the averaging over the initial velocity, the correlation of the parallel velocity becomes

$$< v(t)\, v(t+\tau) > \;=\; \bar{v}_T^2\, e^{-\nu(2t+\tau)} + A \sum_{n=1}^{2} \int_0^t dt_1 e^{-\nu(2t+\tau-2t_1)} \quad (9)$$

or

$$< v_z(t) v_z(t+\tau) >= \left(\bar{v}_T^2 + \frac{A}{2\nu}(e^{2\nu t} - 1) \right) e^{-\nu(2t+\tau)} \quad (10)$$

A stationary stochastic velocity is obtained with:

$$A = 2\nu \bar{v}_T^2 \,. \quad (11)$$

Assuming this choice, we finally get

$$< v(t) v(t+\tau) >= \bar{v}_T^2\; e^{-\nu\tau} \,. \quad (12)$$

Now splitting the guiding centre position in an averaged part and a fluctuating part, $\mathbf{x} =< \mathbf{x} >_{b,\eta} (t) + \delta\mathbf{x}(t)$, and using the initial conditions $< z(t) >_{b,\eta}= z_0$, $< x(t) >_{b,\eta}= x_0$ and $\delta z(0) = 0, \delta x(0) = 0$, the formal solution of the two first equations in (5) is obtained as:

$$\delta z(t) = \int_0^t d\tau\, v(\tau), \quad \delta x(t) = \int_0^t d\tau\, b[z(\tau)]\, v(\tau)\,. \quad (13)$$

Recalling (12), the mean square displacement in the z-direction is now easily obtained :

$$< \delta z^2(t) > \;=\; \int_0^t dt_1 \int_0^t dt_2 \; < v(t_1) v(t_2) > \;=\; \frac{2\, \bar{v}_T^2}{\nu^2} [\nu t - (1 - e^{-\nu t})] \quad (14)$$

The parallel motion in this model goes like t for large times. This motion is therefore diffusive with a diffusion coefficient $D_z = \frac{1}{2}\partial_t < \delta z^2(t) >= \bar{v}_T^2/\nu$. The mean free path being $\lambda = v_T/\nu$, we also have $D_z = \lambda^2\nu$. A quantity of greather interest is the mean square displacement in the "radial" x-direction:

$$< \delta x^2(t) > \;=\; \int_0^t \int_0^t d\tau_1 d\tau_2 \; < b[z(\tau_1)]\, \eta_\parallel(\tau_1)\, b[z(\tau_2)]\, v_z(\tau_2) >_{b,\eta} \quad (15)$$

Using the Fourier representation for $\mathbf{b}$, and recalling (2), the equation becomes

$$< \delta x^2(t) > \;=\; \int_0^t d\tau_1 \int_0^t d\tau_2 \int dk_1 \int dk_2 \times < b_{k_1} b_{k_2}\, e^{[ik_1 z(\tau_1) + ik_2 z(\tau_2)]}\, v_z(\tau_1) v_z(\tau_2) >_{b,\eta} = 2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int dk\, \mathcal{B}(k) \mathcal{Z}(k;\tau_1,\tau_2) \quad (16)$$

where

$$\mathcal{Z}(k;\tau_1,\tau_2) =< e^{ik\int_t^{t+\tau} d\theta v_z(\theta)} v_z(t+\tau) v_z(t) >_\eta \,. \quad (17)$$

With $\tau = t_2 - t_1$, the running diffusion coefficient is obtained in the form:

$$D_x(t) = \int_0^t d\tau \int dk\, \mathcal{B}(k)\, \mathcal{Z}(k;\tau)\,. \quad (18)$$

When $\lim_{t\to\infty} D_x(t)$ is finite and positive the displacement is diffusive; when it is zero the process is subdiffusive; whereas when it is infinite the process is superdiffusive. We have already shown that the parallel displacement is diffusive. The perpendicular motion, driven by the product of two stochastic processes, the parallel velocity and the magnetic fluctuations, shows a fundamentally different behaviour. To get a more explicit expression for the diffusion coefficient, we first rewrite $\mathcal{Z}(k;\tau)$ as

$$\mathcal{Z}(k;\tau) = \frac{1}{k^2} \partial_a \partial_b < e^{ik\int_a^b d\theta v_z(\theta)} > |_{a=t;b=t+\tau}\,. \quad (19)$$

The exponential is expanded leading to

$$\mathcal{Z}(k;\tau) \;=\; \frac{1}{k^2}\partial_a\partial_b \sum_{n=0}^{\infty} \frac{(-k^2)^n}{2n!} \int_a^b d\theta_1 ... \int_a^b d\theta_{2n} \int^{\theta_1} d\tau_1 ... \int^{\theta_{2n}} d\tau_{2n}\; e^{-\nu(\theta_1-\tau_1)} ... e^{-\nu(\theta_{2n}-\tau_{2n})} < \alpha(\tau_1)...\alpha(\tau_{2n}) > |_{a=t;b=t+\tau} \quad (20)$$

Since the acceleration is assumed to be a gaussian process, we have:

$$< \alpha(\tau_1)...\alpha(\tau_{2n}) > \;=\; (2n-1)!! < \alpha(\tau_1)\alpha(\tau_2) > ... < \alpha(\tau_{2n-1})\, \alpha(\tau_{2n}) > \quad (21)$$

With this expression, we recognize in (20) a product of correlations of the parallel displacement. We can thus write:

$$\mathcal{Z}(k;\tau) \;=\; \frac{1}{k^2}\partial_a\partial_b \sum_{n=0}^{\infty} (-k^2 < \delta z^2 >)^n \frac{(2n-1)!!}{2n!} \;=\; \frac{1}{k^2}\partial_a\partial_b \exp\frac{-k^2\psi}{2}|_{a=t;b=t+\tau} \quad (22)$$

Recalling (4) we easily derive the running diffusion coefficient defined in (18) as:

$$D(t) \;=\; \partial_t \int dk\, \frac{\lambda_\parallel\, \beta^2}{\sqrt{2\pi} k^2} \exp\frac{-k^2(< \delta z^2 > + \lambda_\parallel^2)}{2} \;=\; \frac{\lambda_\parallel\, \beta^2}{2}\, \partial_t \sqrt{< \delta z^2 > + \lambda_\parallel^2}\,. \quad (23)$$

The mean square displacement in the parallel direction being linear in t, the running diffusion coefficient in the perpendicular direction is proportional to $t^{-1/2}$ a signature of a subdiffusive process. This is an example of "strange" anomalous behaviour.

II. SOLUTION OF THE KINETIC EQUATION

We now consider the possibility of describing "strange" behaviours starting from a kinetic equation. We thus consider the stochastic Liouville equation (a kinetic equation) whose characteristics are defined from the Langevin equations:

$$\begin{aligned}\frac{\partial}{\partial t}f + \frac{\partial}{\partial z}vf &+ \frac{\partial}{\partial x}vb(z)f \\ &+ \frac{\partial}{\partial v}[-\nu v + \alpha(t)]f = 0\,. \qquad (24)\end{aligned}$$

Here $\{x, y, z, v\}$ are the phase space coordinates. A much simpler equation is however obtained by considering the velocity as a given stochastic process:

$$\frac{\partial}{\partial t}f + \frac{\partial}{\partial z}v(t)f + \frac{\partial}{\partial x}v(t)b(z)f = 0\,. \qquad (25)$$

We assume that the distribution function has an averaged and a fluctuating component:

$$f(x, z, v; t) = F(x, z; t) + \delta f(x, z; t)\,. \qquad (26)$$

Then, the equilibrium is defined by a double averagings: a first one over the velocity the second one over the fluctuating magnetic field. We also assume that $F(x, z; t)$ gives the averaged density (also called profile):

$$F(x, z; t) = n(x, t), \qquad \partial_x n(x) = -X_1 = cst\,. \qquad (27)$$

Applying the averaging procedure, we get:

$$\begin{aligned}\frac{\partial}{\partial t}F(x, z; t) &+ \frac{\partial}{\partial z} << v(t)\,\delta f >> \\ &+\frac{\partial}{\partial x} << v(t)\,b(z)\,\delta f >>= 0 \qquad (28)\\ (\frac{\partial}{\partial t} + v(t)\frac{\partial}{\partial z})\delta f &= v(t)b(z)X_1 \\ &+ \frac{\partial}{\partial z} << v\,\delta f >> \qquad (29)\end{aligned}$$

The equation of the averaged profile has the form of a continuity equation. The components of the particle flux are thus immediately identified as

$$\Gamma_z =<< v(t)\,\delta f >>\,,\ \Gamma_x =<< v(t)\,b(z)\,\delta f >>\ . \qquad (30)$$

The propagator associated to (29) being

$$G(z, t|z', t') = \delta(z - \int_{t'}^{t} d\theta v(\theta) - z')\,, \qquad (31)$$

the solution of the initial value problem takes the form

$$\begin{aligned}\delta f(z; t) &= \int dz' \Big\{ G(z, t|z', 0)\delta f(z', 0) \\ &+ \int d\tau\, G(z, t|z', \tau)\big[v(\tau)b(z')X_1 \\ &+ \frac{\partial}{\partial z'} << v(\tau)\,\delta f(z'; \tau) >>\big]\Big\} \qquad (32)\end{aligned}$$

The first term tends to zero for long times whereas the last term gives a vanishing contribution to the particle flux. Thus, instead of (32), we can use the approximate expression:

$$\delta f(z; t) = \int_0^t d\tau v(\tau) b\big(z - \int_\tau^t d\theta v(\theta)\big)\, X_1\,. \qquad (33)$$

Substituting this expression in the particle flux (30), we get:

$$\begin{aligned}\Gamma_x &= << b(z)v(t) \int_0^t d\tau v(\tau) b\big(z - \int_\tau^t d\theta v(\theta)\big) X_1 >> \\ &= \frac{X_1}{(2\pi)^2} \int_0^t d\tau \int dk_1 \int dk_2\ < b(k_1)b(k_2) > \times \\ &\quad < e^{-ik_1 z - ik_2[z - \int_\tau^t d\theta v(\theta)]} v(t)v(\tau) > \qquad (34)\end{aligned}$$

This equation has the form of a flux-force relation $\Gamma_x = D_x(t)X_1$ where $D_x(t)$ is the running diffusion coefficient

$$D_x(t) = \frac{1}{(2\pi)^2} \int_0^t d\tau \int dk\, \mathcal{B}(k)\, \mathcal{Z}(k, \tau)\,. \qquad (35)$$

This result shows that the Langevin and the kinetic approaches lead to the same running diffusion coefficient. Both methods can equivalently be used to study the subdiffusion of the particles in a stochastic field. They however show different mathematical complexities: in the former, the main difficulty lies in the resummation of the expansions whereas in the later the difficulty comes from the resolution of the kinetic equation. There is a third approach to this problem: the so-called continuous random walk approach [2] which will now be described.

A. CTRW

The subdiffusion (exemplified by the particle diffusion in stochastic magnetic field) or superdiffusion processes are cases of non-Brownian motion. Other examples of anomalous diffusion have been found in numerical studies of one-dimensional maps, in the Chirikov-Taylor standard map, in stochastic webs and in experiments on tracer diffusion in flow systems. The theory which will be described here is based on Levy distributions. It permits to go beyond the Brownian description of the random motion.

In this theory, the whole dynamics is expressed in terms of stochastic events: a particle makes a jump $\mathbf{r}$ of arbitrary length and arbitrary direction at time t, then remains at its new position for a finite time τ, after which it makes a new jump. We assume these jumps are mutually independent. The probability density of a jump of length $\mathbf{r}$ is denoted $f(\mathbf{r})$. We assume that the jumps are performed at random intervals, that must be defined statistically.
We introduce three distribution functions in time:

1) $\psi(t)$ is the waiting time distribution, defined as the probability density of a pause of duration t between two successive steps. The Laplace transform of this function is

$$\begin{aligned} \hat{\psi}(s) &= \int_0^\infty dt e^{-st}\psi(t) \\ \psi(t) &= \frac{1}{2\pi i}\int_\Gamma ds e^{st}\hat{\psi}(s) \end{aligned} \tag{36}$$

Where Γ is the Bromwich contour in the complex plane.

2) $\psi_j(t)$ represents the probability distribution that starting at $t=0$ the j-th step occurs at time t

$$\begin{aligned} \psi_j(t) &= \int_0^t d\tau\psi(t-\tau)\psi_{j-1}(\tau), \qquad j>1 \\ \psi_1(t) &= \psi(t) \end{aligned} \tag{37}$$

or, in Laplace representation:

$$\hat{\psi}_j(s) = \hat{\psi}(s)\hat{\psi}_{j-1}(s), \qquad \hat{\psi}_1(s) = \hat{\psi}(s)\,. \tag{38}$$

Then:

$$\hat{\psi}_j(s) = [\hat{\psi}(s)]^j \equiv \hat{\psi}^j(s)\,; \tag{39}$$

3) $\phi(t)$ is the probability that the particle remains immobile during a time t after a jump. Since the probability that at least one step occurs in the interval $[0-t]$ is $\int_0^t d\tau\,\psi(\tau)$, we have:

$$\phi(t) = 1 - \int_0^t d\tau\psi(\tau) = \int_t^\infty d\tau\psi(\tau)\,. \tag{40}$$

The Laplace transform of $\phi(t)$ is

$$\hat{\phi}(s) = \frac{1}{s} - \frac{1}{s}\hat{\psi}(s)\,. \tag{41}$$

We now consider the spatial evolution and define three functions:

1) $f(\mathbf{x})$ is the probability distribution of a jump of length $\mathbf{x}$;

2) $n_j(\mathbf{x})$ is the probability distribution to be in $\mathbf{x}$ after j steps. It is given by

$$n_j(\mathbf{x}) = \int d\mathbf{x}' f(\mathbf{x}-\mathbf{x}')\,n_{j-1}(\mathbf{x}'),\ \ n_0(\mathbf{x}) = \delta(\mathbf{x}) \tag{42}$$

whose fourier transform is given by

$$\tilde{n}_j(\mathbf{k}) = \tilde{f}(\mathbf{k})\,\tilde{n}_{j-1}(\mathbf{k}), \qquad \tilde{n}_0(\mathbf{x}) = 1 \tag{43}$$

which yields to

$$\tilde{n}_j(\mathbf{k}) = [\tilde{f}(\mathbf{k})]^j \equiv \tilde{f}^j(\mathbf{k})\,; \tag{44}$$

3) $q(\mathbf{x},t)$ is the probability that the particle arrives in $\mathbf{x}$ immediately after a jump. This position can be reached after j-steps. Thus:

$$q(\mathbf{x},t) = \sum_{j=0}^\infty \psi_j(t)\,n_j(\mathbf{x})\,. \tag{45}$$

The Laplace transform of $q(\mathbf{x},t)$ is

$$\hat{q}(\mathbf{x},s) = \sum_{j=0}^\infty \hat{\psi}_j(s)\,n_j(\mathbf{x}) = \sum_{j=0}^\infty \hat{\psi}^j(s)\,n_j(\mathbf{x})\,, \tag{46}$$

and its fourier transform reads

$$\tilde{\hat{q}}(\mathbf{k},s) = \sum_{j=0}^\infty \hat{\psi}_j(s)\,\tilde{n}_j(\mathbf{k}) = \sum_{j=0}^\infty \hat{\psi}^j(s)\,\tilde{f}^j(\mathbf{k})\,. \tag{47}$$

The summation is immediate:

$$\tilde{\hat{q}}(\mathbf{k},s) = \frac{1}{1-\hat{\psi}(s)\tilde{f}(\mathbf{k})}\,. \tag{48}$$

We now want to determine the probability distribution that a particle starting in $\mathbf{x}=0$ at time $t=0$ be in $\mathbf{x}$ at time t (arrival in $\mathbf{x}$ at time τ then stay a time $t-\tau$ for an arbitrary τ). This quantity will be called the density profile. Thus:

$$n(\mathbf{x},t) = \int_0^t d\tau\,\phi(t-\tau)q(\mathbf{x},\tau)\,. \tag{49}$$

With (41) and (48) its Laplace-Fourier transform

$$\tilde{\hat{n}}(\mathbf{k},s) = \hat{\phi}(s)\,\tilde{\hat{q}}(\mathbf{k},s)\,, \tag{50}$$

becomes

$$\tilde{\hat{n}}(\mathbf{k},s) = \frac{1-\hat{\psi}(s)}{s}\,\frac{1}{1-\hat{\psi}(s)\tilde{f}(\mathbf{k})}\,. \tag{51}$$

Transforming back to space and time, we have

$$\begin{aligned} n(\mathbf{x},t) &= \frac{1}{2\pi i}\int_\Gamma ds\,e^{st}\frac{1}{(2\pi)^d}\int d^d\mathbf{k}\,e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &\times \frac{1}{1-\hat{\psi}(s)\tilde{f}(\mathbf{k})} \end{aligned} \tag{52}$$

The Laplace-Fourier transform of the time derivative of $n(\mathbf{x},t)$ is $s\tilde{\hat{n}}(\mathbf{k},s)-1$ or

$$s\,\tilde{\hat{n}}(\mathbf{k},s) - 1 = \frac{1-\hat{\psi}(s)}{1-\hat{\psi}(s)\tilde{f}(\mathbf{k})} - 1 \tag{53}$$

Then, defining the function $\hat{\phi}(s)$

$$\hat{\phi}(s) = \frac{s\hat{\psi}(s)}{1-\hat{\psi}(s)} \tag{54}$$

we get:

$$s\tilde{\hat{n}}(\mathbf{k}, s) - 1 = -\hat{\phi}(s)[1 - \tilde{f}(\mathbf{k})\tilde{\hat{n}}(\mathbf{k}, s) \tag{55}$$

which transformed back to space and time gives

$$\begin{aligned} \partial_t n(\mathbf{x}, t) &= \int_0^t d\tau \phi(t-\tau) \times \\ &\times \left[-n(\mathbf{x}, \tau) + \int d^d\mathbf{k}\, f(\mathbf{x}-\mathbf{x}')n(\mathbf{x}', t)\right] \end{aligned} \tag{56}$$

This generalized master equation was derived by Montroll & Shlessinger in 1984. It is a non-Markovian equation which has two input functions: the transition probability and the waiting time distribution. The "strange" character of the dynamics will be associated, in the next section, to the non analytic properties of these two distributions.

B. Model distributions

Let us now assume that $f(k)$ is analytic near $k = 0$. By definition of the Fourier transform, we have

$$\tilde{f}(\mathbf{k}) = \int d^d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \tag{57}$$

Then, the expansion of the exponential, gives:

$$\tilde{f}(\mathbf{k}) = \int d^d\mathbf{x} \left(1 + i\mathbf{k}\cdot\mathbf{x} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{x})^2 + ...\right) f(\mathbf{x}) \tag{58}$$

Clearly, the two first coefficients are proportional to the two first moments of $f(\mathbf{x})$:

$$\tilde{f}(\mathbf{k}) = 1 + i\mathbf{k}\cdot <\mathbf{x}> -\frac{1}{2}\,\mathbf{kk} :<\mathbf{xx}> +... \tag{59}$$

In the case of an isotropic function, this expression simplifies into

$$\tilde{f}(\mathbf{k}) = 1 + i\mathbf{k}\cdot <\mathbf{x}> -\frac{1}{2d}\,k^2 <r^2> +... \tag{60}$$

We now consider an analytic function of the form:

$$\tilde{f}(\mathbf{k}) = 1 + i\mu\cdot\mathbf{k} - \frac{1}{2d}\sigma^2 k^2 + ..., \qquad k \to 0 \tag{61}$$

and a non-analytic waiting time distribution:

$$\hat{\psi}(s) = 1 - \tau_D^\alpha s^\alpha + ..., \qquad 0 < \alpha < 1, \qquad s \to 0 \tag{62}$$

This waiting time distribution has an infinite first moment: the average of $< t >$ is given by $\partial_s\hat{\psi}(s) \approx s^{\alpha-1}$. Assuming α is comprised in the range $0 < \alpha < 1$ then, in the limit $s \to 0$, we have $< t > \to \infty$. The density profile then result in

$$\tilde{n}(\mathbf{k}, t) = \tau_D^\alpha\, \mathcal{L}_t^{-1}\left[\frac{1}{s^{1-\alpha}[1-\tilde{f}(\mathbf{k})] + \tau_D^\alpha s \tilde{f}(\mathbf{k})}\right] \tag{63}$$

Recalling again the definition of the Fourier transform, we have

$$\begin{aligned} \frac{\partial}{i\,\partial\mathbf{k}}\tilde{n}(\mathbf{k}, t) &= \int d^d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{x}\, n(x), \\ \frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{k}}\tilde{n}(\mathbf{k}, t) &= -\int d^d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, r^2\, n(x), \end{aligned} \tag{64}$$

The mean value of the displacement is obtained as

$$<\mathbf{x}(t)> = \frac{\partial}{i\,\partial\mathbf{k}}\tilde{n}(\mathbf{k}, t)|_{k=0} = \left(\frac{t^\alpha}{\tau_D^\alpha \Gamma(1+\alpha)} - 1\right)\mu \tag{65}$$

whereas the mean square displacement is

$$\begin{aligned} < r^2(t) > &= \frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{k}}\tilde{n}(\mathbf{k}, t)|_{k=0} \\ &= 2\mu^2\left[\frac{t^{2\alpha}}{\tau_D^{\alpha 2}\Gamma(1+2\alpha)} - \frac{2t^\alpha}{\tau_D^\alpha\Gamma(1+\alpha)} + 1\right] \\ &+ \sigma^2\left[\frac{t^\alpha}{\tau_D^\alpha\Gamma(1+\alpha)} - 1\right] \end{aligned} \tag{66}$$

Hence, for long times:

$$< r^2(t) > = 2\mu^2 \frac{t^{2\alpha}}{\tau_D^{\alpha 2}\Gamma(1+2\alpha)}\,. \tag{67}$$

We now see that all behaviours are possible: for $0 < \alpha < 1/2$ the behaviour is subdiffusive, whereas for $1/2 < \alpha < 1$ it is superdiffusive. We note that when $\mu = 0$ the mean square displacement is

$$< r^2(t) > = \frac{1}{\Gamma(1+\alpha)}\sigma^2\left(\frac{t^\alpha}{\tau_D}\right), \qquad 0 < \alpha < 1 \tag{68}$$

In this case the behaviour is always subdiffusive. A direct comparison with the diffusion coefficient of the radial diffusion in the stochastic magnetic field permits to determine the parameters of the distributions. In the case of subdiffusion we have $\alpha = 1/2$. The other parameters [3] are $d = 1$, $\tau_D = \frac{\lambda_\parallel^2}{2D_z}$ and $\sigma^2 = \sqrt{\pi\beta^2\lambda_\parallel^2}$.

C. Applications of the CTRW to generalized Chirikov-Taylor standard map: Tokamap, Guiding centre map

Let us now consider the application of the previous method to maps in which strange diffusion is also observed but where in general no analytical expression of the diffusion coefficient is known. We shall first define a method for transforming systems of differential equations into maps. As a first exemple, we shall consider the Chirikov-Taylor standard map which will be obtained by a discretization of the pendulum Hamiltonian:

$$H(\theta,p)=\frac{p^2}{2}+\frac{A}{2\pi}\cos 2\pi\theta, \tag{69}$$

where the phase space coordinates θ and p satisfy the Poisson bracket relation $[\theta,p]=1$. The equations of motion are simply Hamilton's equations of (69):

$$\frac{dp}{d\tau}\equiv[p,\theta]\,\frac{\partial H}{\partial\theta}=A\sin 2\pi\theta,\quad \frac{d\theta}{d\tau}\equiv[\theta,p]\,\frac{\partial H}{\partial p}=p \tag{70}$$

They are in canonical form i.e. obtained from canonically conjugated coordinates. It is possible to extend the dimensionality of the phase space without changing the dynamics:

$$H(\theta,p,\zeta,t)=\zeta+\frac{p^2}{2}+\frac{A}{2\pi}\cos 2\pi\theta \tag{71}$$

where t can be considered as a time coordinate. We now have two non-vanishing Poisson bracket : $[\theta,p]=1$ and $[t,\zeta]=1$ and in addition to the equations of motion (70) we have:

$$\frac{d\zeta}{d\tau}=0,\qquad \frac{dt}{d\tau}=1. \tag{72}$$

The Hamiltonian (71) can obviously be written as a sum of two contributions:

$$H(\theta,p,\zeta,t)\equiv H_0(p,\zeta)+H_1(\theta) \tag{73}$$

with

$$H_0(p,\zeta)\equiv\zeta+\frac{p^2}{2},\quad H_1(\theta)\equiv\frac{A}{2\pi}\cos 2\pi\theta \tag{74}$$

This Hamiltonian is now easily randomized by interrupting the perturbation regularly in time. We write

$$H(\theta,p,\zeta,t)=H_0(p,\zeta)+H_1(\theta)\sum_{s=1}^{\infty}\Pi(t-sT) \tag{75}$$

where $\Pi(t-sT)$ is a "step function":

$$\begin{aligned}
\Pi(t-sT) &= 0 && t<sT-\frac{\delta\tau}{2}\\
\Pi(t-sT) &= T/\delta\tau && \text{for}\quad sT-\frac{\delta\tau}{2}<t<sT+\frac{\delta\tau}{2}\\
\Pi(t-sT) &= 0 && t>sT+\frac{\delta\tau}{2}
\end{aligned} \tag{76}$$

Here s is an integer, T is the period of the interruption, $\delta\tau$ is the time interval of non-interrupted perturbation. The constraints on $\Pi(t-sT)$ are (i) equal to $T/\delta\tau$ at time $t=sT$ (ii) its derivative should be 0 at time $t=s\alpha$ and, (iii) equal to 0 at times $t=sT+r'\delta\tau$ with r' an integer different from 0. The equations of motion read:

$$\begin{aligned}
\frac{d\theta}{d\tau} &= \frac{\partial H_0(p,\zeta)}{\partial p},\quad \frac{dt}{d\tau}=1,\\
\frac{dp}{d\tau} &= -\frac{\partial H_1(\theta)}{\partial\theta}\sum_{s=1}^{\infty}\Pi(t-sT)\\
\frac{d\zeta}{d\tau} &= -H_1(\theta)\sum_{s=1}^{\infty}\frac{\partial}{\partial t}\Pi(t-sT)
\end{aligned} \tag{77}$$

We now apply a discretization procedure. To this end, we use a peculiar type of canonical transformation called symplectic integration which moves a phase space point (q_{r-1},p_{r-1}) at time t to a point (q_r,p_r) at time $t+d\tau$, following a trajectory of $H(q,p,t)$. In the case of a first-order symplectic integrator, the displacement is given by a set of difference equations:

$$\begin{aligned}
q_r &= q_{r-1}+\Delta\tau\left.\frac{\partial H\,(q,p)}{\partial q}\right|_z\\
p_r &= p_{r-1}-\Delta\tau\left.\frac{\partial H\,(q,p)}{\partial p}\right|_z
\end{aligned} \tag{78}$$

where $z=\{q,p\}$ with $q=\alpha q_r+(1-\alpha)\,q_{r-1}$ and $p=(1-\alpha)\,p_r+\alpha p_{r-1}$ (α is a real number). The parameter $\delta\tau$, usually a small quantity, is the time step of integration. We assume $\delta\tau\leq\Delta\tau<T$. For simplicity we assume that $\delta\tau=\Delta\tau$ and furthermore that $r\Delta\tau=T$ with r an integer. At the $(s\times r)$-th iteration *i.e.* at time t_{sr} :

$$t_{sr}=s\;r\delta\tau=s\,T$$

the function $\Pi(t-s\,T)$ differs from zero

$$\Pi(t_{mn}-s\,T)=\Pi(n\;\alpha-s\,T)=T/\delta\tau$$

and the system is perturbed. For all other dicrete time values we have $\Pi(t-sT)=0$ and the system is unperturbed.

Applying these rules to the r first iterations, we get:

$$\begin{aligned}
\theta_{r+1} &= \theta_i+T\left.\frac{\partial H(\theta,p)}{\partial p}\right|_{\theta=\theta_i;p=p_{r+1}}\\
p_{r+1} &= p_i-T\left.\frac{\partial H_1(\theta)}{\partial\theta}\right|_{\theta=\theta_i;p=p_{r+1}}\\
\zeta_{r+1} &= \zeta_i,\qquad t_{r+1}=T
\end{aligned}$$

where the initial position in the phase space is $\{\theta_i,p_i,\zeta_i,0\}$. The set of difference equations obtained here are very similar to those obtained by a direct application of the symplectic integrator on the continuously perturbed Hamiltonian. The difference lies in the time step $\Delta\tau << 1$ in the later case and $T \approx 1$ in the former. Applying our procedure (with $\alpha = 1$, other forms of the map are obtained by varying α and $T = 1$) on the pendulum Hamiltonian, we obtain the Chirikov Taylor standard map [5] :

$$\begin{aligned} \theta_1 &= \theta_i + \ p_1 \\ p_1 &= p_i + \ A\cos 2\pi\theta_i \end{aligned} \tag{79}$$

The same procedure can be applied to the guiding centre motion in a perturbed magnetic field [4]. Our derivation of the Guiding centre map is based on the equations:

$$\dot{\mathbf{Y}} = \frac{U}{B}\,\frac{1}{1+\rho_\parallel \mathbf{b}\cdot\nabla\times\mathbf{b}}\,\left[\mathbf{B} + \nabla\times\rho_\parallel\mathbf{B}\right] \tag{80}$$

where $\rho_\parallel = U/\Omega_c$, $\Omega_c = eB/mc$ is the Larmor frequency and U is a function of the GC energy $\mathcal{E}$ and of the GC position $\mathbf{Y}$. The main problem encountered here follows from the fact that the guiding centre coordinates are by construction non-canonical. The equations depends on the non-canonical guiding centre position $\mathbf{Y}$, the two invariants $\mathcal{E}$ and M. Recalling $\mathbf{Y}$ is a three dimensional vector, we can form two couples of non-canonical coordinates (one is an invariant) i.e. a $2-D$ Hamiltonian system. We shall use a 2-D set of canonical coordinates denoted by $P_\theta, \theta, P_\zeta, \zeta$ associated by the two non-vanishing Poisson brackets

$$[\zeta, P_\zeta] = 1, \qquad [\theta, P_\theta] = 1 \tag{81}$$

As we are considering a toroidal magnetic confinement, the non-canonical coordinates will be the usual toroidal components of the GC position $\mathbf{Y}$ denoted by ρ, θ, ζ. The fourth coordinate is $\mathcal{E}$. We assume the confining magnetic field can be given in Clebsh representation

$$\mathbf{B} = \nabla\alpha\times\nabla\psi \tag{82}$$

where α is dimensionless quantity and ψ a magnetic flux function but for the application we choose the standard model of magnetic field:

$$\mathbf{B} = \frac{B_0}{h}\,[\mathbf{e}_\zeta + \frac{\rho}{q(\rho)R_0}\mathbf{e}_\theta] \tag{83}$$

where the function h is simply

$$h = 1 + \frac{\rho}{R_0}\cos\theta \tag{84}$$

and $q(\rho)$ is the safety factor. This field is divergenceless and is consistent with an expansion in ϵ_T. The norm of the field is

$$B = \frac{B_0}{h}\,\mathcal{G} \tag{85}$$

with

$$\mathcal{G} = [1 + (\frac{\rho}{q(\rho)R_0})^2]^{1/2} \tag{86}$$

The toroidal scaling factors are $l_\rho = 1$, $l_\theta = \rho$ and $l_\zeta = 1 + (\rho/R_0)\cos\theta$ where R_0 is the distance from the coordinate system ρ, θ to the symmetry axis. The standard magnetic field satisfies a Clebsh representation (82) with $\psi = B_0\,\rho^2/2$ and $\alpha = \int \frac{1}{h}\,d\theta - \zeta\frac{1}{q(\rho)}$. The combination of the drift equations, and of the standard model of magnetic field in toroidal coordinates permit to derive exact canonical coordinates. We assume the magnetic perturbation in toroidal invariant. Then ζ is a cyclic coordinate. The guiding centre Hamiltonian is the usual one

$$H(P_\theta, \theta) \equiv \mathcal{E} = \frac{1}{2}U^2(\rho, \theta, \mathcal{E}) + MB(\rho, \theta) \tag{87}$$

The perturbed magnetic field has three components:

$$B_\theta = \frac{B_0}{h}2\psi\epsilon_T\mathcal{G},\; B_\zeta = \frac{B_0}{h},\; B_\rho = \frac{B_0}{h}\frac{k}{2}\frac{\rho\epsilon_T}{1+\psi}\sin\theta$$

with $\mathcal{G} = q^{-1}(\rho) - k(1+\psi)^{-1}\cos\theta$ and $\psi = \rho^2/2$. The canonical momentum are obtained in the form

$$\begin{aligned} P_\theta &\equiv 2\frac{z\epsilon}{\epsilon_T}\epsilon_T^2\rho_\parallel\frac{\mathcal{G}\psi}{h} + z\int d\rho\frac{\rho}{h} \\ P_\zeta &\equiv \frac{z\epsilon}{\epsilon_T}\rho_\parallel - z\int d\rho\,\mathcal{G} \end{aligned} \tag{88}$$

The guiding centre map is then

$$\begin{aligned} P_{\theta\,1} &= P_{\theta\,0} - 2\pi T\frac{\epsilon_T^2\Omega^2}{\epsilon^2 D}(DR - TS)\sin 2\pi\theta_0 \\ \theta_1 &= \theta_0 + T\frac{\Omega^2}{\epsilon D}S \end{aligned} \tag{89}$$

The functions appearing in the map are

$$\begin{aligned} D &= \frac{\rho}{h}(1+G^2) + \epsilon\rho_\parallel\frac{\partial(G\rho/h)}{\partial\rho} \\ R &= -\rho_\parallel I + \epsilon\frac{U^2+MB}{\Omega^2}\frac{\rho}{h} \\ S &= \frac{U}{\Omega}G + \epsilon\frac{U^2+MB}{\Omega^2}\frac{1}{B}\frac{\partial B}{\partial\rho} \\ T &= \epsilon\rho_\parallel G\frac{\rho^2}{h^2} + k\frac{\rho^2}{h}(\frac{\epsilon_T\rho_\parallel}{(1+\psi)^2 + GI}) \end{aligned} \tag{90}$$

where $I = \int_0^\rho d\rho\frac{\rho}{(1+\psi)^2}$. The map given here reproduces all the features of the toroidal trajectories. In particular it includes the two classes of trajectories: the banana and the passing ones. The map has however unusual features, it is implicit and requires a root finding procedure to solve at each step the equation for ρ given by:

$$P_\theta \equiv \epsilon_T G\frac{\rho}{h}P_\zeta + \frac{\epsilon_T}{\epsilon}(\int_0^\rho d\rho\frac{\rho}{h} + G\frac{\rho}{h}\int_0^\rho d\rho\,\mathcal{G}) \tag{91}$$

To describe field line motion, we remove all the drifts and all variations of the parallel velocity. This is accomplished by taking $M = 0$ and by keeping only the dominant terms in ϵ_T then $P_\theta = \epsilon_T \psi/\epsilon$. In order to go from the time dependence of the trajectory to the ζ-dependence of the field line it is necessary to divide all the equations by the equation of motion for ζ. Applying the discretization procedure, we get ($T = 1$)

$$\begin{aligned} \theta_1 &= \theta_0 + \{q^{-1}(\psi_1) - k\frac{1}{(1+\psi_1)^2}\cos 2\pi\theta_0\} \\ \psi_1 &= \psi_0 - 2\pi k\frac{\psi_1}{1+\psi_1}\sin 2\pi\theta_0 \end{aligned} \tag{92}$$

which is nothing else than the Tokamap [9].

We now quote some recent results obtained with two of the maps derived here: the standard map (79) and the tokamap (92). We first consider the standard map on which many detailed studies have been performed [5]. The momentum diffusion as a function of the perturbation amplitude A has been very well reproduced analytically by a Fourier path integral method [6]. However, important discrepencies are observed for "almost" integer values of A [7]. These discrepencies are attributed to set of accelerator modes. For those particular values of the amplitude, island of stability emerge which attract an orbit and keep it for some times in a laminar phase (an exemple of intermittency). The CTRW method has been applied and compared with the statistical properties of $10^7 - 10^8$ trajectories in the accelerator mode domain. It has been observed that a good fit of the density defined by $n(r,t) \approx f(\xi)/t^{1/\gamma}$ where ξ is the scaling variable $\xi = |r|/t^{1/\gamma}$ and where $f(\xi)$ follows a Levy distribution

$$\begin{aligned} f(\xi) &= \exp(-c\xi^2), \quad \xi < 1 \\ &= \xi^{-\gamma-1}, \quad 1 < \xi \quad \text{and}\, |r| < t \\ &= 0, \quad |r| > t \end{aligned} \tag{93}$$

is obtained when $\gamma = 5/3$ in the domain of superdiffusion.

More recently, the dynamic in the stochastic layer of the standard map was described by symbolic dynamics [8]. The value of A was in that case $A = 0.7$. For this value, the phase portrait shows a main island surrounded by a stochastic layer bounded above and below by undestroyed KAM barriers. An orbit oscillates between three basins, one surrounding the whole island, one above the island and the third one below the island. The CTRW theory was applied to analyse the properties of a single trajectory followed over a very long time interval. In this subdiffusion regime a very good agreement could also be obtained.

Although superdiffusion in the field line map (the Tokamap 92) has not yet been observed, it is obvious that the CTRW theory also applies to this map in the subdiffusion regime [9]. However, not all situations have been considered: new possibilities could arise since the tokamap, depending on the shear profile, also belong to the category of non twist map which have typical features like reconnections [10]. Such processes are considered in [9] where it is shown that the tokamap in the reverse shear configuration presents robust regular core region while the edge is stochastic. Related recent works, oriented toward the analysis of the dynamics of the field line structure in the DED, are in progress [12].

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