Magnetic fields and plasmas

T.J. Schep

Eindhoven University of Technology,
5600 MB Eindhoven, The Netherlands

1 Introduction

Plasmas and magnetic fields are inseparably related in numerous physical circumstances. This is not only the case in natural ocurring plasmas like the solar corona and the earth magnetic tail, but also in laboratory plasmas like tokamaks and stellarators.
A high-temperature plasma can not be contained by the material walls of a vessel but only by strong magnetic fields. The resulting Lorentz forces will tie the particles to the magnetic field and will force them to follow the field lines. To a first approximation plasma confinement means confinement of magnetic field lines. Such confined fields are necessarily curved and inhomogeneous and provide the geometrical shape of the plasma. They also dominate its physical behaviour.
The cross-section of Coulomb collisions is a strongly decreasing function of the energy of the interacting particles. Therefore, the mean free paths of charged particles in a high-temperature fusion device are very long and the particles will trace out their trajectories over the spatial size of the confining magnetic field before they are swept out of their orbits by collisions. Therefore, collisional dissipation will be very weak in these high-temperature plasmas. In the limit of zero dissipation, a number of plasma quantities will be conserved during the motion of the system. An important conservation law in this ideal limit is the conservation of magnetic flux.

Magnetic field lines and the conservation of magnetic fluxes are the subjects of this lecture. Since magnetic fields are globally divergence-free, their spatial structure can be described in terms of field lines, which are curves in space that are everywhere tangent to the field and whose density represents the strength of the magnetic field. However, this description of the geometrical structure of the field holds only at any instant of time. The question is if we can associate with the magnetic field a velocity field such that the time evolution of the magnetic field can be described in terms of a continuous motion of field lines.
This concept of a flux conserving flow of field lines will be discussed in the next Section. It will be shown that this concept breaks down at magnetic nulls, points or lines where the magnetic field vanishes, and at closed magnetic field lines. At these geometrical positions flux conservation can be violated and field lines can loose their identity. There, reconnection processes can take place which change the global topology of the magnetic field. Magnetic nulls where the total field vanishes, do occur in space plasmas like the geomagnetic tail and possibly also in solar coronal loops. In fusion devices with strong externally applied magnetic fields, like tokamaks and stellarators, such 3D nulls do not occur. In the approximation of closed and nested magnetic flux surfaces, however, closed magnetic field lines are dense in these systems.
Flux conserving flows of field lines will be discussed in the next Section. This concept has been treated by Newcomb [1]; a concise review has been given by Greene [2]. In Section 3 we will discuss a few properties of magnetic nulls. Analogously to particle orbits in phase space, the spatial pattern of magnetic field lines can be described as a Hamiltonian system, where the Hamiltonian contains all the information about the topology of the field lines. This subject,which has been extensively treated by Boozer [3-5], will be discussed in Section 4, where a canonical representation of magnetic fields will be introduced. This representation is illustrated in Section 5 with the example of an axisymmetric tokamak configuration with closed magnetic surfaces. Magnetic islands are treated in Section 6. In Section 7 we will return to the time dependent fields of Section 2 and deal with flux conservation from a different point of view. Finally, we will touch upon the dynamics of a plasma in Sections 8-10, where we will discuss some implications of Ohm's law.

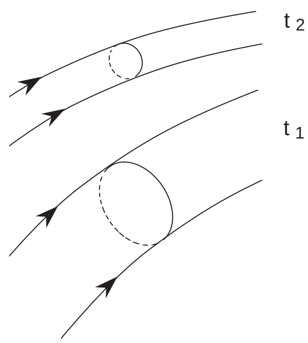

Figure 1: Motion of line bundle.

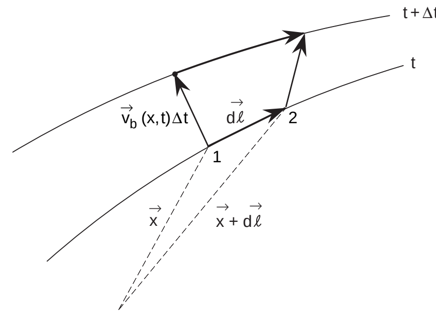

Figure 2: A line element moving with the flow $\vec{v}_b(\vec{x}, t)$.

2 Flux conserving flows

In plasma physics one often speaks about magnetic fields and their behaviour in time in terms of field lines and their motion. In this Section we discuss the reasons of this parlance and the conditions under which it is justified. The basic problem is whether the time evolution of a magnetic field can be described by a flux conserving motion of field lines.
Since magnetic fields are globally divergence-free, their spatial structure can be conveniently discussed in terms of their field lines, which are curves in space that are everywhere tangent to the magnetic field vector and whose density represents the strength of the magnetic field. A field line $\vec{x}(s)$ is the solution of

$$\frac{d\vec{x}}{ds} = \alpha\vec{B}, \tag{1}$$

where the variable s parametrizes the field line and $\alpha ds = dl/B$, where dl is the differential length along the line.
Analogously to particle trajectories in phase space, the spatial pattern of magnetic field lines can be described as a Hamiltonian flow, where the Hamiltonian contains all the information about the topology of the field lines.
However, this description only refers to the geometrical structure of the field and only holds at any separate instant of time. The question arises if the time evolution of the magnetic field can be described in terms of a continuous motion of line bundles. This requires that we associate a velocity with the magnetic field such that we can identify a field line from one instant to the next and label it unambiguously. Consider a smooth, geometric (kinematic) velocity field $\vec{v}_b(\vec{x}, t)$. Geometrical lines, surfaces and volumes move locally with this velocity. This velocity need not to be a physical quantity and need not to have a direct relationship with the dynamical velocity of the medium (plasma). It should be stressed that the magnetic field and the associated electric field are determined by the *dynamics* of the physical system and are assumed to be given. Here, we deal only with *geometrical and kinematical* aspects.
During the motion the lines will be stretched and the cross-section of the bundle will change, as is sketched in Fig. 1.
The field lines will keep their identity if the magnetic field remains tangent to the lines and if the number of field lines crossing a surface, i.e., if the magnetic flux is conserved with time. Then the time evolution of the magnetic field can be visualized as a flux conserving motion of field lines. The change with time of a line element $\vec{dl}$ that moves with the field $\vec{v}_b$ (see Fig. 2) is determined by

$$\frac{d}{dt}\vec{dl} = \vec{dl} \cdot (\nabla\vec{v}_b). \tag{2}$$

It follows that the time rate of change of a comoving surface element $\vec{dS} = \vec{dl}_1 \times \vec{dl}_2$ is,

$$\frac{d}{dt}\vec{dS} = -(\nabla\vec{v}_b) \cdot \vec{dS} + (\nabla \cdot \vec{v}_b)\vec{dS}. \tag{3}$$

This implies that the time rate of change of the magnetic flux through a surface element is

$$\frac{d}{dt}\vec{B} \cdot \vec{dS} = (\frac{\partial\vec{B}}{\partial t} - \nabla \times \vec{v}_b \times \vec{B}) \cdot \vec{dS}. \tag{4}$$

It is concluded that the conservation of flux through any surface element requires that a smooth velocity field $\vec{v}_b(\vec{x}, t)$ can be defined such that the given magnetic field $\vec{B}(\vec{x}, t)$ satisfies [1,2]

$$\frac{\partial\vec{B}}{\partial t} = \nabla \times \vec{v}_b \times \vec{B}. \tag{5}$$

This equation also implies that if initially the magnetic field is tangent to a line element, this remains so in the course of time. *Hence, the time evolution of the magnetic field can be visualized as a flux conserving flow of field lines if Eq. (5) is valid.*

According to Faradays equation, $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$, the validity of (5) requires that the electric field can be written in the form

$$\vec{E} = -\vec{v}_b \times \vec{B} - \nabla F, \tag{6}$$

where $F(\vec{x}, t)$ is an arbitrary, single-valued function. This equation determines both $\vec{v}_b(\vec{x}, t)$ and $F(\vec{x}, t)$ given the field $\vec{B}(\vec{x}, t)$. The component of Eq. (6) parallel to the field $\vec{B}$

$$\vec{E} \cdot \vec{B} = -\vec{B} \cdot \nabla F \tag{7}$$

determines F. Then, the field line velocity is given by

$$\vec{v}_b = (\vec{E} + \nabla F) \times \vec{B}/B^2. \tag{8}$$

The component of the velocity along the magnetic field does not play a role and may be disregarded.
Different choices for F yield different flow velocities. If two velocities and functions F are related by $(\vec{v}_{b1} - \vec{v}_{b2}) \times \vec{B} + \nabla(F_1 - F_2) = 0$ and $\vec{B} \cdot \nabla(F_1 - F_2) = 0$, then either line flow represents the evolution of the field. This arbitrariness indicates that $\vec{v}_b$ is not a physical quantity.
The line velocity (8) is well-defined except when F is undetermined, and possibly at magnetic nulls where the field strength B vanishes. Magnetic nulls where the total field vanishes do occur in space plasmas like the geomagnetic tail and solar flares. They do not occur in fusion devices with strong externally applied magnetic fields like tokamaks and stellarators.
Equation (7), which determines F, is called a magnetic differential equation [6, 7]. Although it can always be solved locally, its full solution depends on boundary conditions and, thus, on global plasma properties. In particular, Eq. (7) can only be solved on a *closed magnetic field line if the loop voltage*

$$\oint \vec{dl} \cdot \vec{E} \tag{9}$$

along the line vanishes . This is a global, not a local condition. If the loop voltage along any closed field line vanishes, then $\vec{v}_b$ is well-defined everywhere and magnetic fluxes are conserved. However, this is a strong requirement that will be satisfied only in ideal cases.

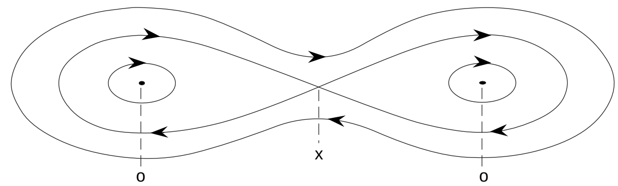

Figure 3: The magnetic field of two parallel, diffuse currents.

In general, the loop integral of the electric field generated by arbitrary plasma motions will not be zero. This means that flux conservation can break down on closed magnetic field lines. Note that the magnetic field strength does not appear in (9), only the geometry of the closed field line plays a role. A well-known 2D configuration with magnetic nulls is the magnetic field of two parallel, diffuse currents. The geometry is sketched in Fig. 3. The hyperbolic null is called an X-point, the elliptic nulls are O-points. The configuration consists of three flux bundles. The two field lines that start and end at the X-point separate these bundles. They are called separatrices. A constant magnetic field in the third direction (perpendicular to the page) could be added to the configuration. Then, if the system is periodic in this direction, the $X-$ and $O-$points in the figure represent field lines that are effectively closed. Magnetic fluxes are conserved if the loop integrals of the electric field along these closed curves vanish.
The existence of magnetic nulls and closed field lines is intimately related with magnetic reconnection and, thus, with changes in field topology. Magnetic reconnection is a field evolution in which some flux tubes flow into or emerge from a point or a line where either the total field strength or some component of the field vanishes. In this process the flux tubes loose their global identity. Loosely speaking one may say that the flux tubes are cut and reconnected in a different order. This local phenomenon changes the global topology. This reconnection process is illustrated in Fig. 4. Flux bundles carrying opposite fluxes flow into an X-point where the magnetic field vanishes. There they loose their identity. Reconnected flux tubes that emerge from this process have a different topology. The figure also illustrates the singular aspect of a reconnection process. Field lines that become adjacent at one position may be far apart elsewhere in space. This means that the process connects regions in space that were originally isolated from each other. This may have as a consequence that large flows of particles and energy are set up along field lines so that transport is greatly enhanced.

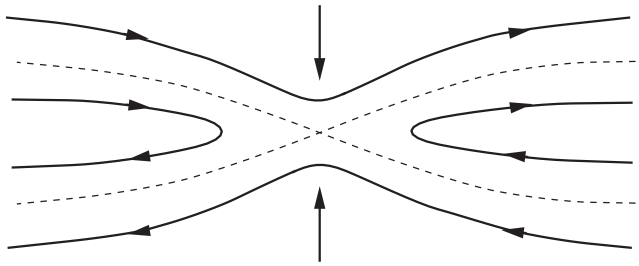

Figure 4: Standard representation of reconnection.

3 Magnetic field lines near a null

The characteristic properties of the field $\vec{B} = B_i \vec{e}_i$ in the neighborhood of a simple magnetic null at $\vec{x} = 0$ can be understood from the linearized expansion around the critical point at $\vec{x} = 0$:

$$B_i = B_{ij} x^j, \qquad B_{ij} = \left(\frac{\partial B_i}{\partial x^j} \right)_{\vec{x}=0}. \tag{10}$$

This expansion yields a constant current density, $\vec{J} = \nabla \times \vec{B}$, in the neighborhood of the null, and, thus, is only valid for fields with a regular current distribution.
Since the magnetic field is divergence-free, the matrix $\{B_{ij}\}$ has zero trace and the sum of its eigenvalues λ_i vanishes. The product of the eigenvalues is equal to the determinant, $D_n \equiv det\ \{B_{ij}\}$. If the determinant does not vanish, the magnetic nulls are isolated points.
Although the velocity $\vec{v}_b$ of a field line may not be regular at a magnetic null (see Eq. (8)), the velocity $\vec{v}_n$ of the null itself remains finite. From (10) one finds that $\vec{v}_n$ is given by [2a]

$$\frac{\partial}{\partial\, t} B_i \Big|_{\vec{x}=0} = -\vec{v}_n \cdot \nabla B_i \Big|_{\vec{x}=0} = -v_{nj} B_{ij} \Big|_{\vec{x}=0}, \tag{11}$$

so that $\vec{v}_n$ is finite if the determinant of the matrix $\{B_{ij}\}$ does not vanish.
The magnetic field lines near a null are obtained by combining Eqs (1) and (10)

$$\frac{dx^i}{ds} = \alpha B_{ij} x^j. \tag{12}$$

These field line equations are invariant under $x^i \to -x^i$ and under $B_{ij} \to -B_{ij},\ s \to -s$.
The matrix $\{B_{ij}\}$ can be written as the sum of a symmetrical and of an anti-symmetrical part. The symmetrical matrix $\{B^s_{ij}\}$ with elements $(B_{ij} + B_{ji})/2$ represents a (local) vacuum field that is produced by currents at large distances. This matrix can always be diagonalized by rotations. The anti-symmetrical part $B^a_{ij} = (B_{ij} - B_{ji})/2$ is related with the local, constant current density $\vec{J}$,

$$J_i = \epsilon_{ijk} \frac{\partial B_k}{\partial x^j} = \epsilon_{ijk} B^a_{kj}, \tag{13}$$

where ϵ_{ijk} are the completely antisymmetric Levi-Civita symbols. In the absence of currents, the matrix $\{B_{ij}\}$ is symmetric and has real eigenvalues λ_l, $(l = 1, 2, 3)$. Hence, complex eigenvalues require a finite current density.

In the coordinate system in which the symmetric part of $\{B_{ij}\}$ is diagonal, we have

$$\vec{B} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -J_3 & J_2 \\ +J_3 & 0 & J_1 \\ -J_2 & -J_1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{14}$$

with $\Sigma a_i = 0$.
In three dimensions, the traceless, symmetric matrix has two independent parameters. The anti-symmetric part contains three independent parameters that determine the orientation of the current with respect to the vacuum field. Hence, five parameters characterize the field in the neighborhood of a magnetic null. In two dimensions, this number is reduced to two: one eigenvalue and the value of the current density in the ignorable direction.

The field $\vec{B}$ can also be written as follows. Since the matrix $\{B_{ij}\}$ has at least one real eigenvalue, we can choose the z-axis in the direction of the corresponding eigenvector. By a subsequent, suitably chosen rotation in the (x, y)-plane, one can write [10]

$$\vec{B} = \begin{pmatrix} a_{11} & -\frac{1}{2} J_z & 0 \\ \frac{1}{2} J_z & a_{22} & 0 \\ -J_y & J_x & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{15}$$

with $\Sigma a_{ii} = 0$. This matrix has the eigenvalues

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}(a_{11} + a_{22}) \pm \frac{1}{2} [(a_{11} - a_{22})^2 - J_z^2]^{\frac{1}{2}}, \\ \lambda_3 &= a_{33} (= -a_{11} - a_{22}). \end{aligned} \tag{16}$$

Complex eigenvalues occur when the current density is sufficiently large.
Two equal eigenvalues occur either if $(\lambda_1 = \lambda_2)$

$$(a_{11} - a_{22})^2 - J_z^2 = 0, \tag{17}$$

or if $(\lambda_3 = \lambda_{1,2})$

$$(a_{11} - a_{22})^2 - J_z^2 = 9 a_{33}^2. \tag{18}$$

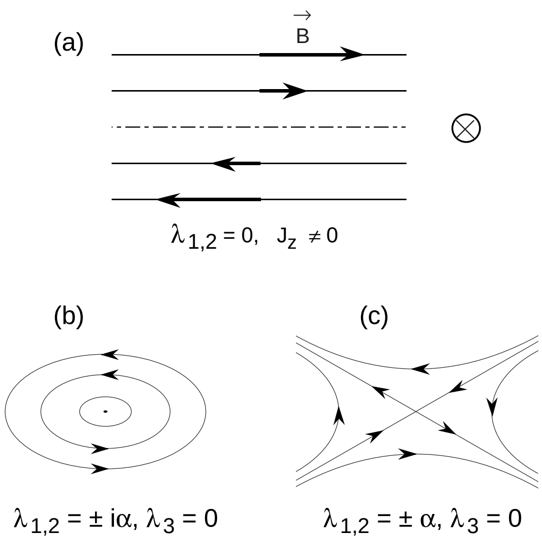

Figure 5: (a) 2D magnetic field configurations, (a) neutral layer: $\lambda_{1,2} = 0, \ D_n = 0$; (b) O-point: $\lambda_{1,2} = \pm i\lambda, \ D_n > 0$; (c) X-point: $\lambda_{1,2} = \pm\lambda, \ D_n < 0$.

The case of two dimensions is described by Eq. (14) with $a_1 = -a_2, \ J_1 = J_2 = 0$. The eigenvalues of the matrix $\{B_{ij}\}$ are

$$\lambda_{1,2} = \pm\frac{1}{2}[4a_1^2 - J_3^2]^{\frac{1}{2}}. \tag{19}$$

When both eigenvalues vanish, the geometry is that of a current layer where the magnetic field vanishes at a neutral line. When the eigenvalues are purely imaginary, the field lines are wrapped around an elliptic point which is called an O-point. This can be verified by solving Eq.(12). When both are real, and thus of opposite sign, the field lines run into and out of a hyperbolic point which is called an X-point. Field lines that start or end at the null are called separatrices. These three cases are illustrated in Fig. 5. Only the latter case can exist with locally a vacuum magnetic field.

A constant magnetic field in the third direction (perpendicular to the page) could be added in each of these examples. Then, if the system is periodic in this direction, the $X-$ and $O-$points in the figures represent field lines that are effectively closed. Each point of the neutral line is the intersection of the plane with such a line. According to what has been said in the previous section, magnetic fluxes are conserved if the loop integrals of the electric field along these closed curves vanish.

An example of a three dimensional null is given by Eq.(14) with $J_{1,2} = 0$. The eigenvalues of the matrix B_{ij} are

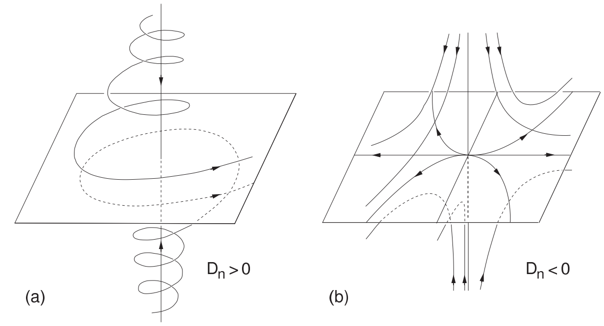

Figure 6: (a) 3-D O-point; (b) 3-D X-point.

$$\lambda_{1,2} = \frac{1}{2}(a_1 + a_2) \pm \frac{1}{2}[(a_1 - a_2)^2 - J_3^2]^{1/2}, \quad \lambda_3 = a_3. \tag{20}$$

Another example is $a_1 = a_3, J_2 = 0$. In this case the eigenvalues are

$$\lambda_1 = a_1, \quad \lambda_{2,3} = -\frac{1}{2}a_1 \pm \frac{1}{2}[9a_1^2 - (J_1^2 + J_3^2)]^{\frac{1}{2}}. \tag{21}$$

Examples of 3-D nulls [8] are presented in Fig. 6.

4 Static equilibria with a magnetic null

According to (10) and (13), the components of the Lorentz force are

$$\vec{J} \times \vec{B}|_i = \epsilon_{ijk} J_j B_k = 2B^a_{ik} B_{kl} x^l. \tag{22}$$

The curl of this force is

$$\nabla \times \vec{J} \times \vec{B}|_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \vec{J} \times \vec{B}|_k = 2\epsilon_{ijk} B^a_{kl} B^s_{lj} \tag{23}$$

since $\epsilon_{ijk} B^a_{kl} B^a_{lj} = 0$. Note that $\vec{B}$ and $\vec{J}$ are both divergence-free so that $\nabla \times \vec{J} \times \vec{B} = [\vec{B}, \ \vec{J}]$, where the Lie bracket is defined as $[\vec{B}, \ \vec{J}] \equiv \vec{B} \cdot \nabla \vec{J} - \vec{J} \cdot \nabla \vec{B}$.

According to Frobenius theorem [9], the current density $\vec{J}$ and the magnetic field $\vec{B}$ span locally a set of (magnetic) surfaces if

$$[\vec{B}, \ \vec{J}] = \alpha_B \vec{B} + \alpha_J \vec{J}, \tag{24}$$

where α_B and α_J are real constants. This condition can be written as

$$2\epsilon_{ijk}B^a_{kl}B^s_{lj} = \alpha_B B_{ij}x^j + \alpha_J\epsilon_{ijk}B^a_{kj}. \qquad (25)$$

To leading order in powers of x one obtains (α_B involves higher order terms in the current)

$$B^a_{kl}B^s_{lj} + B^s_{kl}B^a_{lj} = \alpha_J B^a_{kj}. \qquad (26)$$

With (14) one finds

$$\begin{pmatrix} 0 & (a_3+\alpha_J)J_3 & -(a_2+\alpha_J)J_2 \\ -(a_3+\alpha_J)J_3 & 0 & (a_1+\alpha_J)J_1 \\ (a_2+\alpha_J)J_2 & -(a_1+\alpha_J)J_1 & 0 \end{pmatrix} = 0. \qquad (27)$$

The important cases with $\alpha_J \neq 0$ are,

$$a_3 = -\alpha_J, \quad a_{1,2}+\alpha_J \neq 0; \quad J_{1,2} = 0, \quad J_3 \ arbitrary,$$

$$a_1 = a_3 = -\alpha_J, \quad a_2 = 2\alpha_J; \quad J_2 = 0, \quad J_{1,3} \ arbitrary. \qquad (28)$$

These cases correspond to Eq.(20) with $a_3 = -\alpha_J$ and to Eq.(21) with $a_1 = -\alpha_J$, respectively. We conclude that the current density and the magnetic field near an isolated null span locally a set of surfaces if and only if Eq. (27) is satisfied.

In the case of a scalar pressure equilibrium, $\vec{J}\times\vec{B} = \nabla p$, the right-hand side of (23) has to vanish

$$B^a_{kl}B^s_{lj} + B^s_{kl}B^a_{lj} = 0. \qquad (29)$$

This implies that the product matrix $B_{kl}B_{lj} = B^a_{kl}B^a_{lj} + B^s_{kl}B^s_{lj}$ is symmetric and has real eigenvalues λ^2 (the squares of the eigenvalues of the matrix B_{ij}). It follows that *the eigenvalues λ_i are either all real or all purely imaginary.*
In 3D, the latter case implies that one eigenvalue vanishes, so that the determinant vanishes. This situation is structurally unstable and can be ignored. This can be seen as follows. Suppose $\vec{B}$ has a zero at $\vec{x} = 0$. Then, adding a small magnetic field $\vec{B}_1$, the null will shift to the position $\delta\vec{x}$ which is given by $[B_{ij}(0) + B_{1ij}(0)]\delta x^j = -B_{1i}(0)$. A solution for $\delta\vec{x}$ exist if the determinant of the matrix in the left hand side does not vanish, i.e. if this matrix has *no* zero eigenvalues.
Equation (27) reads in the scalar pressure limit,

$$\begin{pmatrix} 0 & a_3J_3 & -a_2J_2 \\ -a_3J_3 & 0 & a_1J_1 \\ a_2J_2 & -a_1J_1 & 0 \end{pmatrix} = 0. \qquad (30)$$

It follows that if some a_i has a finite value than the corresponding current density J_i must vanish. There are only two null configurations that are consistent with a static equilibrium with isotropic pressure. All current components may have arbitrary values if all a_i's vanish, i.e. if there is no vacuum magnetic field. The other case is when the current density has only a single component, say $J_3 \neq 0, J_{1,2} = 0$. Then $a_3 = 0$ and $a_1 = -a_2$. This corresponds to the 2D magnetic null or neutral line of Fig.5. The two conjugate eigenvalues are given by Eq.(19), the third eigenvalue vanishes.

5 Field line Hamiltonian

In this Section the Hamiltonian character of magnetic field lines is clarified. Since a magnetic field is globally divergence-free, it possesses a single-valued vector potential $\vec{A}$. In an arbitrary coordinate system (ρ,θ,ϕ), $\vec{A}$ can be represented as

$$\vec{A} = \chi\nabla\theta - \psi\nabla\phi + \nabla G, \qquad (31)$$

where $\chi(\vec{x}), \psi(\vec{x}), \theta(\vec{x}), \phi(\vec{x})$ and $G(\vec{x})$ are continuous functions of space and may depend on parameters like the time.
The gauge function G does not play a role in the representation of the magnetic field $\vec{B} = \nabla\times\vec{A}$,

$$\vec{B} = \nabla\chi\times\nabla\theta + \nabla\phi\times\nabla\psi. \qquad (32)$$

Hence, a divergence-free field can be characterized by two fluxes, $\chi(\vec{x})$ and $\psi(\vec{x})$. We limit the discussion to configurations with strong magnetic fields and assume that the 3-coordinate ϕ can be chosen such that the contravariant component $B^\phi = \vec{B}\cdot\nabla\phi = \nabla\chi\times\nabla\theta\cdot\nabla\phi$ never vanishes in the volume under consideration. In a toroidal geometry with a strong toroidal field, e.g., an obvious choice for ϕ is the toroidal angle. The non-vanishing of B^ϕ implies that the functions $\chi(\vec{x})$, $\theta(\vec{x})$ and $\phi(\vec{x})$ can be inverted, $\vec{x} = \vec{x}(\chi,\theta,\phi)$ because the Jacobian of the transformation $(\vec{x}) \rightarrow (\chi,\theta,\phi)$,

$$D = \frac{\partial\vec{x}}{\partial\chi}\times\frac{\partial\vec{x}}{\partial\theta}\cdot\frac{\partial\vec{x}}{\partial\phi} = \frac{1}{\nabla\chi\times\nabla\theta\cdot\nabla\phi} = \frac{1}{\vec{B}\cdot\nabla\phi}, \qquad (33)$$

is finite and never vanishes. On the basis of Eq. (33) we may take (χ,θ,ϕ) as a set of coordinates. The function

$\psi(\vec{x})$ can be expressed in terms of the new variables, $\psi(\vec{x}) = \psi_H(\chi, \theta, \phi)$.
The basis vectors associated with the coordinates (χ, θ, ψ), are $\vec{e}_\chi = \partial\vec{x}/\partial\chi$, $\vec{e}_\theta = \partial\vec{x}/\partial\theta$ and $\vec{e}_\phi = \partial\vec{x}/\partial\phi$, the cobasis vectors are $\vec{e}^\chi = \nabla\chi$, $\vec{e}^\theta = \nabla\theta$ and $\vec{e}^\phi = \nabla\phi$. In what follows we will frequently use the relations $\vec{e}^i \cdot \vec{e}_j = \delta^i{}_j$ where $\delta^i{}_j$ is the Kronecker symbol.

In case of a toroidal field geometry, the coordinates (χ, θ, ϕ) can be chosen as follows. Such a device is periodic in the poloidal (the short way around the torus) and in the toroidal direction (the long way around the torus). If θ and ϕ denote the corresponding angle-like variables, then the magnetic flux through a surface ϕ = constant is

$$\int \vec{B} \cdot \vec{dS}_\phi = \int \vec{B} \cdot \nabla\phi \; D d\chi d\theta = \int d\chi d\theta. \qquad (34)$$

Since the system is 2π-periodic in θ, $2\pi\chi$ is the flux in the ϕ-direction that is enclosed by a surface χ = constant.

A magnetic field line is an integral curve $\vec{x}(s)$ that is the solution of Eq.(1). Along the field line we have $d\chi = d\vec{x} \cdot \nabla\chi = \alpha ds \vec{B} \cdot \vec{\nabla}\chi$ etc., so that Eq. (1) can be written as

$$\alpha ds = \frac{d\chi}{\vec{B} \cdot \nabla\chi} = \frac{d\theta}{\vec{B} \cdot \nabla\theta} = \frac{d\phi}{\vec{B} \cdot \nabla\phi}. \qquad (35)$$

Since $\vec{B} \cdot \nabla\phi \neq 0$ everywhere, we may choose $s = \phi$ and $\alpha = (B \cdot \nabla\phi)^{-1}$. Then, using expression (32) for the magnetic field, the field line equations (35) take the form [3],

$$\frac{d\chi}{d\phi} = -\frac{\partial\psi_H}{\partial\theta} \quad , \quad \frac{d\theta}{d\phi} = \frac{\partial\psi_H}{\partial\chi}. \qquad (36)$$

These are Hamilton equations in two-dimensional phase space (χ, θ). *The flux function* $\psi_H(\chi, \theta, \phi)$ *is the Hamiltonian.* It contains all the information on the topology of the field lines. The conjugate variables (χ, θ) play the roles of the generalized momentum and coordinate. The parameter ϕ along the field line has the role of the "time"-variable. Equation (36) can be used for field line tracing. These field lines may be either regular or stochastic. When the Hamiltonian ψ_H is given, then the integration of Eq. (36) yields χ and θ as functions of ϕ. Subsequently, the transformation equations $\vec{x} = \vec{x}(\chi, \theta, \phi)$ give the field lines $\vec{x}(\phi)$ in space. On the basis of Eq. (36), expression (32) is called a *canonical* representation of the magnetic field.

The field line equations can also be obtained from the variational principle

$$\delta \int \vec{A} \cdot d\vec{x} = \delta \int d\phi \; \vec{A} \cdot \frac{d\vec{x}}{d\phi} = 0 \qquad (37)$$

with $\vec{A} \cdot \delta\vec{x} = 0$ at the end points of the interval of integration. The substitution of expression (31) for the vector potential brings Eq. (37) in the form

$$\delta \int d\phi \; [\chi \frac{d\theta}{d\phi} - \psi_H] = 0. \qquad (38)$$

This is Hamilton's principle. After performing the variations with χ and θ as independent variables, the field line equations (36) are recovered.
The Hamiltonian structure of the field line geometry gives access to the wide body of Hamilton theory, in particular to the well-known apparatus of canonical transformations.
The variational principle (37) does not depend on the gauge function G in (31). Therefore this function can be used to generate canonical transformations $(\chi, \theta) \rightarrow (\chi_1, \theta_1)$ and $\psi_H \rightarrow \psi_{H_1}$. In terms of the new coordinates (χ_1, θ_1) and the new Hamiltonian $\psi_{H_1}(\chi_1, \theta_1, \phi)$, the field line equations (36), Hamilton's principle (38), and the magnetic field representation (32) are unchanged. The ϕ-coordinate is the "time" variable and remains unchanged. Well-known examples of functions that generate canonical transformations [10] are $f_1(\theta, \theta_1, \phi)$ and $f_2(\chi_1, \theta, \phi)$ with

$$\frac{\partial f_1}{\partial\theta_1} = -\chi_1 \quad , \quad \frac{\partial f_1}{\partial\theta} = \chi \quad , \quad \psi_{H_1} = \psi_H + \partial f_1/\partial\phi,$$

and

$$\frac{\partial f_2}{\partial\theta} = \chi \quad , \quad \frac{\partial f_2}{\partial\chi_1} = \theta_1 \quad , \quad \psi_{H_1} = \psi_H + \partial f_2/\partial\phi,$$

respectively. Indeed, it can be shown that any gauge transformation is a canonical transformation and vice versa [4].

A case of particular interest arises when a canonical transformation exists such that the new Hamiltonian only depends on the variable $\chi, \psi_H = \psi_H(\chi)$. In this case, the field line equations (36) can be integrated and yield,

$$\theta - \iota\phi = \theta_0 \quad , \quad \chi = \chi_0, \qquad (39)$$

where θ_0 and χ_0 are constants, and

$$\iota(\chi) \equiv \frac{d\psi_H}{d\chi} \qquad (40)$$

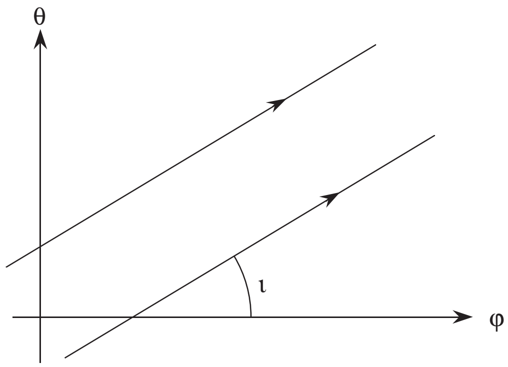

Figure 7: Straight field lines in the integrable case.

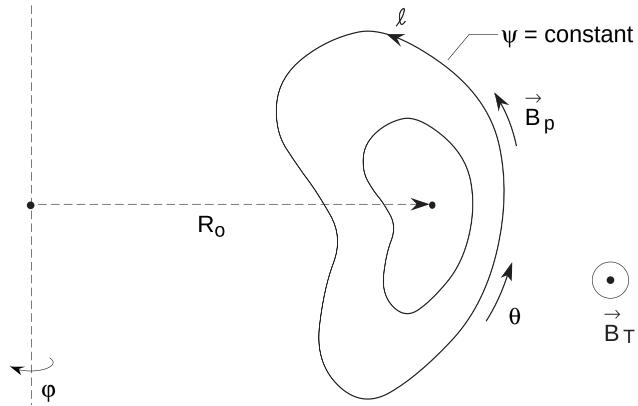

Figure 8: Cross section of a flux surface with a plane through the axis of symmetry.

is the rotational transform. Hence, in this integrable case the field lines are straight lines on surfaces $\chi =$ constant, which now coincide with surfaces $\psi =$ constant. The angle between a field line and the ϕ-axis is determined by $\iota(\chi)$. In the tokamak literature, usually the inverse of the rotational transform, the magnetic winding number q $= \iota^{-1}$, is used. When ι is constant in space, i.e. when it has the same value on all magnetic surfaces $\psi =$ constant, then all field lines are parallel. When ι is space-dependent, field lines on neighbouring surfaces have different directions. This is called *magnetic shear.*

6 Axisymmetric tokamak

We illustrate the discussion of the preceding section with the important example of an axisymmetric tokamak with closed magnetic surfaces.
The magnetic field in such a configuration can be represented as

$$\vec{B} = \vec{B}_T + \vec{B}_p = RB_T\nabla\phi + \nabla\phi \times \nabla\psi, \qquad (41)$$

where B_T and B_p are the toroidal and poloidal field strengths, respectively, ϕ is the toroidal angle and R the distance to the axis of symmetry. Both contributions to $\vec{B}$ are divergence-free. Due to axisymmetry, $\vec{B} \cdot \nabla\psi = 0$ so that the magnetic field lines lie on surfaces $\psi =$ constant, which are assumed to be closed and nested. The inner most surface is the magnetic axis. The poloidal field is tangent to a cross-section l of the flux surface with a plane through the axis of symmetry. The poloidal flux between two neighbouring flux surfaces ψ and $\psi + d\psi$ is ($dl_\phi = d\phi/|\nabla\phi|,\ dl_\psi = d\psi/|\nabla\psi|$)

$$\oint \vec{B}_p \cdot \vec{dl}_\phi \times \vec{dl}_\psi = \oint \frac{B_p}{|\nabla\psi||\nabla\phi|} d\phi d\psi = 2\pi d\psi. \qquad (42)$$

Hence, the field line Hamiltonian of the previous Section is just the poloidal flux function ψ. The toroidal flux between these surfaces is

$$2\pi d\chi = \oint \vec{B}_T \cdot \vec{dl}_\psi \times \vec{dl} = d\psi \oint \frac{B_T}{RB_p} dl, \qquad (43)$$

where dl is the element of length along the curve l. From Eq. (43) it follows that the rotational transform can be expressed as

$$\iota^{-1} = \frac{d\chi}{d\psi} = \frac{1}{2\pi} \oint \frac{B_T}{RB_p} dl. \qquad (44)$$

It remains to construct a poloidal variable θ such that the magnetic field lines are straight in the $\theta - \phi$ plane. The field lines are given by

$$\frac{d\theta}{d\phi} = \frac{\vec{B} \cdot \nabla\theta}{\vec{B} \cdot \nabla\phi}.$$

These lines are straight if $d\theta/d\phi = q^{-1}(\psi)$ is independent of θ. Now $\vec{B} \cdot \nabla\phi = B_T/R$ and $\vec{B} \cdot \nabla\theta = B_p\ \partial\theta/\partial l$, so that the definition

$$\theta = q^{-1} \int_0^l dl \frac{B_T}{RB_p} \qquad (45)$$

guarantees that the field lines are straight. The system is periodic in the poloidal variable θ. If we choose this period to be 2π, we see that $q = \iota^{-1}$ according to Eqs (44) and (45). The straight field lines are given by $\phi - q\theta =$ constant. Rational flux surfaces are surfaces on which q is a rational number, $q = m/n$. On these surfaces the field lines are closed: after encircling the

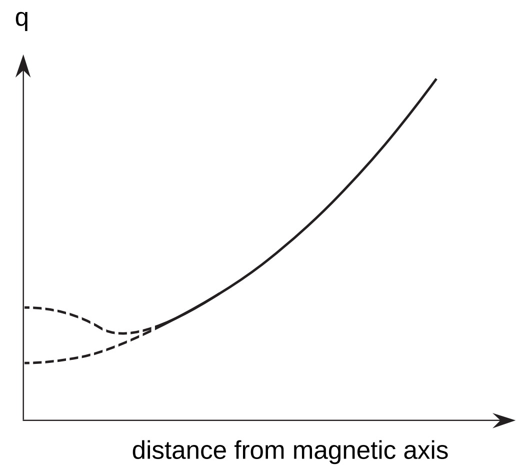

Figure 9: q-profile in a tokamak.

torus n times in the poloidal direction and m times in the toroidal direction, a field line returns to its original point. When q is irrational, the field line fills the flux surfaces ergodically.
In a tokamak, q is positive and increases monotonically with the distance from the magnetic axis, with the possible exception of the centre of the plasma. Since rational numbers are dense, it follows that *magnetic surfaces with closed magnetic field lines are dense in a tokamak.*

7 Magnetic islands

In this section we consider magnetic perturbations of the toroidal magnetic geometry with closed flux surfaces that was discussed in the previous section. The magnetic field is given by Eq. (32). The poloidal flux function ψ (the ϕ-Hamiltonian)

$$\psi = \bar{\psi}(\chi) + \tilde{\psi}(\chi, \theta, \phi) \tag{46}$$

is the sum of an averaged part $\bar{\psi}(\chi)$ related to the background field with closed magnetic surfaces and of a perturbation $\tilde{\psi}$ that is periodic in (θ, ϕ) and may depend on time. The rotational transform of the background field is $d\bar{\psi}/d\chi = \iota(\chi)$.
We introduce the helical flux function ψ_{he} and the coordinate λ with respect to a flux surface $\chi = \chi_0$ of the background field where the rotational transform $\iota(\chi_0)$ equals a rational number ι_0,

$$\psi_{he} = \psi - \iota_0 \chi \quad , \quad \lambda = \theta - \iota_0 \phi. \tag{47}$$

(According to Section 4, this corresponds to the canonical transformation $(\chi, \theta) \rightarrow (\chi, \lambda)$ with generating function $f_2 = \chi(\theta - \iota_0 \phi)$. The new Hamiltonian is $\psi_{he} = \psi + \partial f_2/\partial \phi$.)
Then, the magnetic field (32) can be written as,

$$\vec{B} = \vec{B}_{he} + \vec{B}_* = \nabla\chi \times \nabla\lambda + \nabla\phi \times \nabla\psi_{he}. \tag{48}$$

Note that both fields $\vec{B}_{he}$ and $\vec{B}_*$ are separately divergence-free.
The curves $\lambda =$ constant form a set of helical lines. On the surface $\chi = \chi_0$ where $\iota = \iota_0$, these helical lines coincide with the field lines of the background field. The first term in Eq. (48) is the helical field $\vec{B}_{he} = \nabla\chi \times \nabla\lambda$. It is tangent to the lines $\lambda =$ constant and has constant rotational transform, so that all its field lines are straight, parallel and closed. The second term in Eq. (48) can be written as

$$\vec{B}_* = \nabla\phi \times \nabla\bar{\psi}_{he} + \nabla\phi \times \nabla\tilde{\psi}, \tag{49}$$

with $\bar{\psi}_{he} = \bar{\psi} - \iota_0 \chi$. The first contribution to Eq. (49) represents the shear field component $\bar{\vec{B}}_*$ of the background field. It vanishes at the rational surface $\iota = \iota_0$ where it changes sign if ι is a monotonic function of χ. This shear field geometry corresponds to the 2D case of Fig.5a.
Suppose that the perturbed flux $\tilde{\psi}$ consists of a single Fourier harmonic,

$$\psi_{he} = \bar{\psi}_{he}(\chi) + \tilde{\psi}_{mn}(\chi)\cos(m\theta - n\phi). \tag{50}$$

It follows that,

$$\vec{B} \cdot \nabla\psi_{he} = \vec{B}_{he} \cdot \nabla\tilde{\psi} = -\frac{m}{D}(\iota_0 - \frac{n}{m})\tilde{\psi}_{mn}\sin(m\theta - n\phi). \tag{51}$$

Hence, perturbations with (m, n)-numbers such that their ratio equals the rational number ι_0, $\iota_0 = n/m$, are resonant with the background magnetic field at the surface $\chi = \chi_0$. In this case we have $\vec{B} \cdot \nabla\psi_h e = 0$ so that the magnetic field lies on surfaces $\psi_{he} =$ constant.

The critical points (χ_c, λ_c) of the configuration are the nulls of the $\vec{B}_*$ field which follow from by $\nabla\psi_{he} = 0$,

$$\frac{\partial\bar{\psi}_{he}}{\partial\chi_c} + \frac{\partial\tilde{\psi}}{\partial\chi_c}\cos m\lambda_c = 0, \quad \tilde{\psi}_{mn}(\chi_c)\sin m\lambda_c = 0. \tag{52}$$

In these points $\vec{B}_*$ vanishes and the magnetic field lines are closed. The nature of these critical points is determined by the second derivatives of ψ_{he}, i.e., by the

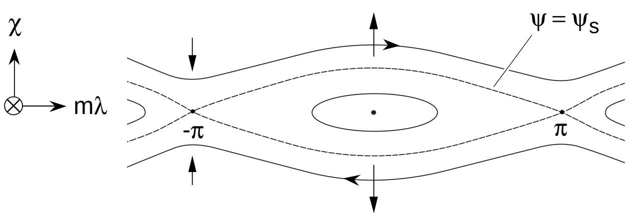

Figure 10: Curves $\psi_{he} = constant$ of a single helicity magnetic island.

properties of the matrix $\nabla\vec{B}_*|_c$ taken at the nulls of $\vec{B}_*$. The simplest case arises when $\tilde{\psi}_{mn}$ is approximately constant near the rational surface. This case is sketched in Fig. 10. Taking $\tilde{\psi}_{mn} > 0$ and $d\iota/d\chi < 0$, then the critical points $(\chi_c = \chi_0, m\lambda_c = \pm\pi)$ are hyperbolic (X-points). The critical points $(\chi_c = \chi_0, m\lambda_c = 0)$ are elliptic (0-points).
The $\vec{B}_*$-lines that connect X-points are finite and start and end at the hyperbolic nulls of $\vec{B}_*$. These curves are called separatrices, they separate the flux bundles inside and outside the magnetic island.
The value ψ_s of ψ_{he} on the separatrix is, according to Eq. (50), $\psi_s = \psi_{he}(\chi_0, m\lambda = \pm\pi) = \bar{\psi}_{he}(\chi_0) - \tilde{\psi}_{mn}$. The separatrix is given by

$$-\tilde{\psi}_{mn} = \frac{1}{2}(\chi-\chi_0)^2\frac{d\iota}{d\chi_0} + \tilde{\psi}_{mn}\cos m\lambda, \qquad (53)$$

where we have expanded $\bar{\psi}_{he}(\chi)$ in Eq. (50) around the rational surface $\chi = \chi_0$. This is valid for small islands. The half-width of the island is

$$\Delta\chi = \left[-\frac{4\tilde{\psi}_{mn}}{d\iota/d\chi_0}\right]^{1/2}. \qquad (54)$$

If we may approximate our toroidal geometry by that of a periodic cylinder with axial flux $\chi = \frac{1}{2}r^2B_T$, B_T being the toroidal field, then the half-width of the island expressed in the cylindrical radius r is

$$\frac{\Delta r}{r_0} = \left[\frac{4q_0\tilde{\psi}_{mn}}{{r_0}^2B_T\hat{s}}\right]^{1/2}, \qquad (55)$$

where r_0 is the position of the resonant surface where $q_0 = m/n$, and $\hat{s} = r_0{q_0}^{-1}dq/dr_0$ is the value of the shear parameter at this surface.
As we have argued in Section 5, rational flux surfaces are dense in a tokamak. Magnetic perturbations with nearby (m,n)-numbers will be resonant on nearby rational surfaces. The distance δr between the rational surfaces $(m+1)/n$ and m/n follows from $(m+1)/n - m/n = \delta r dq/dr_0$. If the amplitudes of these perturbations are such that the sum of the half-widths of their islands becomes of the order of the distance δr between their rational surfaces, islands will start to overlap. Magnetic field lines may then pass from one island structure to the other and wander stochastically through the volume. This will give rise to enhanced radial transport of particles, energy, and momentum.

8 Time-dependent fields

In Sections 3-6 we have limited the discussion to the spatial aspects of the field structure. There the time occurred only as a parameter. In this Section we consider the time dependence explicitly and take the vector potential $\vec{A}$, the gauge G and the variables χ and θ to be functions of $\vec{x}$ and of time t. The ϕ-Hamiltonian is $\psi_H(\chi,\theta,\phi,t)(=\psi(\vec{x},t))$ and the transformation equations are $\vec{x} = \vec{x}(\chi,\theta,\phi,t)$. Again, mainly geometrical aspects will be discussed and the time dependence of all quantities is assumed to be given. Dynamical aspects in the form of Ohm's law are introduced at the end of this Section.
Upon differentiating the vector potential (31) partial to t, we obtain

$$\left(\frac{\partial\vec{A}}{\partial t}\right)_{\vec{x}} = \frac{\partial\chi}{\partial t}\nabla\theta - \frac{\partial\theta}{\partial t}\nabla\chi - \left(\frac{\partial\psi_H}{\partial t} + \frac{\partial\psi_H}{\partial\chi}\frac{\partial\chi}{\partial t} + \frac{\partial\psi_H}{\partial\theta}\frac{\partial\theta}{\partial t}\right)\nabla\phi + \nabla S, \qquad (56)$$

where

$$S = \left(\frac{\partial G}{\partial t}\right)_{\vec{x}} + \chi\frac{\partial\theta}{\partial t}. \qquad (57)$$

From the identity $(\partial\vec{x}/\partial t)_{\vec{x}} = 0$ and the transformation equations $\vec{x} = \vec{x}(\chi,\theta,\phi,t)$ follows

$$\frac{\partial\vec{x}}{\partial t} = -\frac{\partial\vec{x}}{\partial\chi}\frac{\partial\chi}{\partial t} - \frac{\partial\vec{x}}{\partial\theta}\frac{\partial\theta}{\partial t} = -\vec{e}_\chi\frac{\partial\chi}{\partial t} - \vec{e}_\theta\frac{\partial\theta}{\partial t}. \qquad (58)$$

The velocity $\partial\vec{x}/\partial t$ is a coordinate velocity; the variables (χ,θ) are constant in the comoving frame

$$\frac{d}{dt}(\chi,\theta) = \left(\frac{\partial}{\partial t} + \frac{\partial\vec{x}}{\partial t}\cdot\nabla\right)(\chi,\theta) = 0. \qquad (59)$$

The $\vec{e}_\chi$ and $\vec{e}_\theta$ components of (58) are

$$\frac{\partial\chi}{\partial t} = \left(\frac{\partial\vec{A}}{\partial t}\right)_{\vec{x}}\cdot\frac{\partial\vec{x}}{\partial\theta} - \frac{\partial S}{\partial\theta},$$
$$\frac{\partial\theta}{\partial t} = -\left(\frac{\partial\vec{A}}{\partial t}\right)_{\vec{x}}\cdot\frac{\partial\vec{x}}{\partial\chi} + \frac{\partial S}{\partial\chi}. \qquad (60)$$

It follows that in general the time evolution of the coordinates (χ, θ) does not have a Hamiltonian structure.
In terms of the electric field $\vec{E} = -\partial\vec{A}/\partial t - \nabla\Phi$ and the coordinate velocity (58), Eq. (56) can be written in the form [4],

$$\vec{E} + \frac{\partial\vec{x}}{\partial t} \times \vec{B} = \frac{\partial\psi_H}{\partial t}\nabla\phi - \nabla(S + \Phi). \qquad (61)$$

Note that $\partial\psi_H/\partial t = d\psi_H/dt$ is the time rate of change of the poloidal flux function in the local frame that moves with the coordinate velocity (58).
Although Eq. (61) and Eq. (6) of Sec. 2 look similar, they are quite different. In Sec. 2 we required flux conservation from the start, which led to Eq.(6). Here, we have introduced the well-defined velocity (58) and did not require flux conservation.
According to Eq. (61), the loop integral of the electric field along a closed field line is

$$\oint \vec{E} \cdot \vec{dl} = \oint \frac{dl}{DB}\frac{\partial\psi_H}{\partial t} = \oint d\phi \frac{\partial\psi_H}{\partial t}. \qquad (62)$$

It is seen that the loop voltage is directly related with the time rate of change of the poloidal flux enclosed by the loop. This loop voltage vanishes when $\partial\psi_H/\partial t$ is periodic in ϕ with zero mean. *Then, Eq. (61) is equivalent to Eq. (6) and, within the freedom discussed below Eq. (9), the field line velocity (8) may be identified with the coordinate velocity (58).*
Ohm's law in resistive magnetohydrodynamics (MHD) reads

$$\vec{E} + \vec{v} \times \vec{B} = \eta\vec{J}. \qquad (63)$$

Here, $\vec{v}$ is the dynamical fluid velocity. The plasma resistivity is denoted by η and $\vec{J}$ is the current density.
In ideal MHD, the resistive term on the right is neglected. In this case $\vec{E} \cdot \vec{B} = 0$ and the loop voltage along a closed field line vanishes. As we have seen this implies flux conservation. The component of Eq. (61) in the direction of the magnetic field is

$$\frac{1}{D}\frac{\partial\psi_H}{\partial t} = \vec{B} \cdot \nabla(S + \Phi). \qquad (64)$$

Hence, in ideal MHD, the choice $S = -\Phi$ makes the ϕ-Hamiltonian Ψ_H independent of time.
However, the integral of the electric field along closed magnetic loops will be finite for general plasma motions and field variations. According to Eq. (62), the poloidal flux will not be conserved in those cases. The resistive Ohm's law (63) yields

$$\oint \vec{E} \cdot \vec{dl} = \oint \eta J_{\|} dl. \qquad (65)$$

If the loop voltage does not vanish, then the right-hand side of this expression must be finite. Hence, if we take the limit of zero resistivity $\eta \to 0$, in order to reobtain ideal MHD, we find that the current density becomes infinite, $J_{\|} \to \infty$. From the point of view of ideal MHD, closed magnetic field lines are singular lines where arbitrary large currents can be generated. This generation of large currents along closed magnetic loops is related with the singular character of reconnection.

9 Ohms law

In this Section we go beyond the geometrical aspects and touch briefly upon the dynamics of the plasma. However, we will limit the discussion to Ohm's law, which was introduced in the preceding section.
In Sections 2 and 8 it has been shown that flux conservation depends on the vanishing of the loop voltage along closed field lines. Only the parallel component of the electric field plays a role in this condition. Arbitrary forces in the direction perpendicular to the magnetic field may be present, e.g. perpendicular resistivity, without violating flux conservation. This can be shown directly as follows.
Let us assume that the plasma satisfies Ohm's law in the form

$$\vec{E} + \vec{v} \times \vec{B} = \vec{F}_{\perp} + \nabla\Phi, \qquad (66)$$

where $\vec{v}$ is the fluid velocity, Φ is a single-valued function, and $\vec{F}_{\perp}$ is an arbitrary, resistive or turbulent force perpendicular to the magnetic field.
Introduce the velocity $\vec{u}$:

$$\vec{u} = \vec{v} + \frac{1}{B^2}\vec{B} \times \vec{F}_{\perp}. \qquad (67)$$

This velocity is well-defined in systems without magnetic nulls. Then, Ohm's law (66) can be written as

$$\vec{E} + \vec{u} \times \vec{B} = \nabla\Phi. \qquad (68)$$

Hence, *the magnetic flux $\int \vec{B} \cdot \vec{dS}$ through any surface that moves with the velocity field $\vec{u}$ is conserved.* Notice that nevertheless magnetic field and plasma are decoupled: the line velocity $\vec{u}$ does not coincide with the fluid velocity $\vec{v}$. This does not mean that the magnetic field is confined; it says that reconnection can not take place

and that the magnetic topology is conserved. It is concluded that perpendicular, turbulent forces associated with turbulent diffusion across the magnetic field, do not violate flux conservation [12].
If turbulent (anomalous) transport is due to the violation of flux conservation (e.g. stochastization of the magnetic field), then turbulent forces $\vec{F}_{\|}$ along the magnetic field have to play a role.

Let us reconsider ideal Ohm's law,

$$\vec{E} + \vec{v} \times \vec{B} = \nabla \Phi. \tag{69}$$

Expressing the electromagnetic fields in terms of the potentials $\vec{A}$ and ϕ, ($\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t,\ \vec{B} = \nabla \times \vec{A}$), this equation reads

$$\frac{\partial}{\partial t}\vec{A} - \vec{v} \times \nabla \times \vec{A} = -\nabla(\phi + \Phi). \tag{70}$$

Upon combing this relation with the expression (2) for the time rate of change of a line element that moves with the fluid velocity $\vec{v}$, one obtains

$$\frac{d}{dt}\vec{A} \cdot \vec{dl} = \vec{dl} \cdot \nabla(\vec{v} \cdot \vec{A} - \phi - \Phi). \tag{71}$$

Since the expression on the right involves the gradient of a single-valued function, it follows that *the loop integral of the vector potential along any closed curve that moves with the fluid, is conserved,*

$$\frac{d}{dt} \oint \vec{A} \cdot \vec{dl} = 0. \tag{72}$$

This is of course equivalent to the by now familiar flux conservation, $\int \vec{B} \cdot d\vec{S} = constant$.

Equation (70) together with the version of Ohms law $\partial\vec{B}/\partial t - \nabla \times \vec{v} \times \vec{B} = 0$, leads to the following equation for the density $\vec{A} \cdot \vec{B}$ of the *magnetic helicity*,

$$\frac{\partial}{\partial t}\vec{A}\cdot\vec{B} + \nabla\cdot\left[(\vec{A}\cdot\vec{B})\vec{v}\right] = \nabla\cdot[(-\phi-\Phi+\vec{v}\cdot\vec{A})\vec{B}]. \tag{73}$$

The right hand side vanishes after integration of this expression over a *closed magnetic flux tube*, so that it follows that *the magnetic helicity in any closed flux tube is a constant of the motion*

$$\frac{d}{dt} \oint_{D_B} d^3x\ \vec{A} \cdot \vec{B} = 0. \tag{74}$$

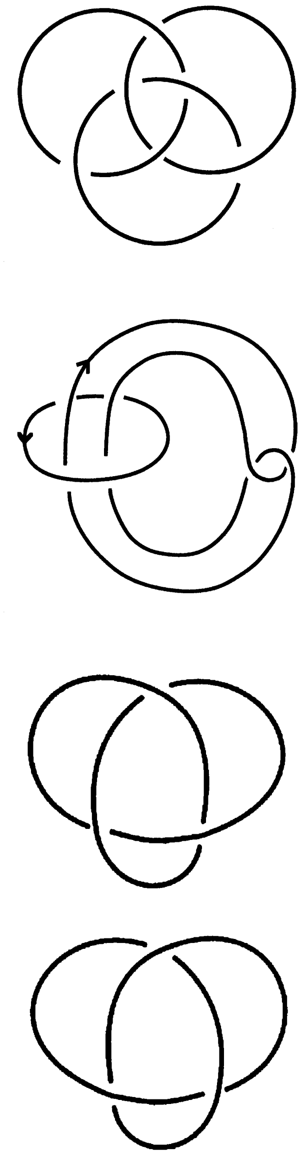

Figure11: *Examples of three linked flux tubes (Borromeo rings), two unlinked tubes, and of knotted flux tubes.*

Note that if one chooses the function f in the gauge transformation $\vec{A} \rightarrow \vec{A} + \nabla f$, $\phi \rightarrow \phi - \partial f \partial t$, such that

$$\frac{\partial}{\partial t} f + \vec{v} \cdot \nabla f = \phi + \Phi - \vec{v} \cdot \vec{A}, \tag{75}$$

then the quantities $\vec{A} \cdot \vec{dl}$ and $\vec{A} \cdot \vec{B} d^3x$ are *local* invariants.

The magnetic helicity invariant $H = \oint_{D_B}\ \vec{A} \cdot \vec{B} d^3x$

of a magnetic configuration has a topological interpretation and finds an explanation in terms of knots and links of entangled fux tubes [13,14]. Figure 11 gives an example of three linked flux tubes (Borromeo rings), two unlinked tubes, and of knotted flux tubes.

10 Generalized Ohm's law

Because of the large difference in mass between the electrons and ions, the electrons will dominate the response of the plasma to electric fields. This means that Ohm's law is basically the electron momentum balance,

$$\frac{\partial}{\partial t}\gamma\vec{v}+\vec{v}\cdot\nabla\gamma\vec{v} \ = -\frac{e}{m}\left(\vec{E}+\vec{v}\times\vec{B}\right)-\nabla F, \qquad (76)$$

where m is the electron mass, e the charge, γ the relativistic mass factor, $\vec{v}$ the velocity of the electron fluid, and F is some function.
In terms of the *generalized fluid momentum*:

$$\vec{P}=\gamma\vec{v}-e\vec{A}/m, \qquad (77)$$

Eq.(76) reads

$$\frac{\partial\vec{P}}{\partial t}-\vec{v}\times\nabla\times\vec{P}=0-\nabla(\frac{e}{m}\phi+\gamma)-\nabla F. \qquad (78)$$

Taking the curl of this equation and introducing the *generalized vorticity*

$$\vec{\Omega}\equiv\nabla\times\vec{P}=\nabla\times(\gamma\vec{v}-e\vec{B}/m), \qquad (79)$$

we obtain the electron momentum balance in the form

$$\frac{\partial\vec{\Omega}}{\partial t}-\nabla\times\vec{v}\times\vec{\Omega}=0. \qquad (80)$$

Upon comparing expressions (78) and (80) with (70) and (5), it is seen that the *loop integral of the generalized momemtum along any comoving curve and the flux of generalized vorticity through any comoving surface* are conserved

$$\oint\vec{P}\cdot\vec{dl}=constant, \qquad \int\vec{\Omega}\cdot\vec{dS}=constant. \qquad (81)$$

Here, comoving means moving with *the electron fluid. These conservation laws imply that collisionless reconnection can occur. Although the generalized vorticity is conserved, the magnetic flux need not to be conserved, but may be converted into electron momentum.*
In Section 8 we have argued that in ideal plasmas ($\eta\to 0$), closed magnetic field lines are singular lines where arbitrary large currents can be generated. Electron inertia will limit these currents that tend to become infinitely large for $m_e\to 0$. Electron inertia sets a new scale-length: the inertia skin-depth $d_e=c/\omega_p$, ω_p being the plasma frequency.
Equations (78) and (80) also imply that the integral of the generalized helicity over a closed $\vec{\Omega}$-flux tube is conserved

$$\oint_{D_\Omega} d^3x\ \vec{P}\cdot\vec{\Omega}\ = constant. \qquad (82)$$

This generalized helicity has the same topological interpretation as magnetic helicity, it is the linking number of $\vec{\Omega}$-lines. *The conservation laws (81) and (82) arise from the electron motion, ions do not play a role.*

The invariants for Euler fluids are recovered from (81) and (82) in the limit $\vec{B}\to 0$,

$$\oint\vec{v}\cdot\vec{dl}, \qquad \int\vec{\omega}\cdot\vec{dS}, \qquad \oint_{D_\omega} d^3x\ \vec{v}\cdot\vec{\omega}, \qquad (83)$$

where the fluid vorticity is defined by $\vec{\omega}\equiv\nabla\times\vec{v}$.
The magnetic conservation laws are obtained in the limit of zero (electron) inertia, $m_e\to 0$:

$$\oint\vec{A}\cdot\vec{dl}, \qquad \int\vec{B}\cdot\vec{dS}, \qquad \oint_{D_B} d^3x\ \vec{A}\cdot\vec{B}. \qquad (84)$$

This generalized vorticity field moves with (is frozen into) *the electron fluid.*
This conservation law implies that collisionless reconnection can occur. Although the generalized vorticity is conserved, magnetic flux may be converted into electron momentum!
In Section 8 we have argued that in ideal plasmas ($\eta\to 0$), arbitrary large currents can be generated along closed magnetic field lines. Electron inertia will limit these currents that tend to become infinitely large for $m_e\to 0$. Electron inertia sets a new scale-length: the inertia skin-depth $d_e=c/\omega_p$, ω_p being the plasma frequency. On smaller scales, fluid models for electrons might break down.

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