Introduction to MHD instabilities

J.P. Goedbloed

FOM-Institute for Plasma Physics 'Rijnhuizen', Association Euratom-FOM
P.O.Box 1207, 3430 BE Nieuwegein, The Netherlands
&
Astronomical Institute, Utrecht University

1 Alfvén wave dynamics

In this paper we will discuss the theoretical framework of ideal MHD stability. We first show that the Alfvén wave is of central importance for the understanding of macroscopic plasma dynamics and, hence, of MHD stability. In section 2 we introduce the elements of spectral theory needed for our study, and in section 3 we apply these methods to inhomogeneous plasmas in cylinder geometry. Conclusions are drawn in section 4.

1) The most robust part of plasma dynamics consists of *the Alfvén wave characteristics* which *are point perturbations* (represented by two dots in the Friedrich's diagram Fig. 1b) propagating along the magnetic field lines. Hence, the Alfvén waves sample the magnetic field structure in a way no other plasma disturbance can do.

2) For homogeneous plasmas, the linear MHD waves are described by the equation of motion involving the force operator $\mathbf{F}$ (discussed more extensively in Sec. 2) in terms of the plasma displacement vector field $\boldsymbol{\xi}$:

$$\mathbf{F}(\boldsymbol{\xi}) = \gamma p_0 \nabla\nabla\cdot\boldsymbol{\xi} + \mathbf{B}_0 \times (\nabla\times(\nabla\times(\mathbf{B}_0\times\boldsymbol{\xi}))) = \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}\,. \quad (1)$$

The first term represents the acoustic (compressible) part and the second one the magnetic (anisotropic) part. Since Alfvén waves are incompressible, only the second term plays a role to leading order. Choosing the magnetic field along z, and also restricting the propagation to that direction, the wave equation for Alfvén waves [1] becomes

$$\frac{B_0^2}{\mu_0\rho_0}\frac{\partial^2\xi_y}{\partial z^2} = \frac{\partial^2\xi_y}{\partial t^2}\,. \quad (2)$$

Substituting plane wave solutions $\exp\{i(k_z - \omega t)\}$, the dispersion equation for Alfvén waves is obtained:

$$\omega_A^2 = k_\parallel^2 v_A^2\,, \qquad v_A \equiv \frac{B_0}{\sqrt{\mu_0\rho_0}}\,, \quad (3)$$

where v_A is the Alfvén velocity.

3) The Alfvén wave frequency vanishes for $k_\parallel = 0$. This implies that the Alfvén waves become marginally stable when the $\mathbf{B}\cdot\nabla$ operator, which represents the *field line bending*, vanishes. This explains why this operator is so important in stability and Alfvén wave dynamics.

4) Of course, the more complete picture, including pressure effects, also involves the two magnetosonic waves (fast and slow). The most concise representation of the three MHD waves is the Friedrichs phase and group diagrams (Fig. 1), which represent the distance travelled from the origin after a certain time interval by plane wave and point disturbances. The group diagram of the Alfvén waves just consists of two dots, as mentioned.

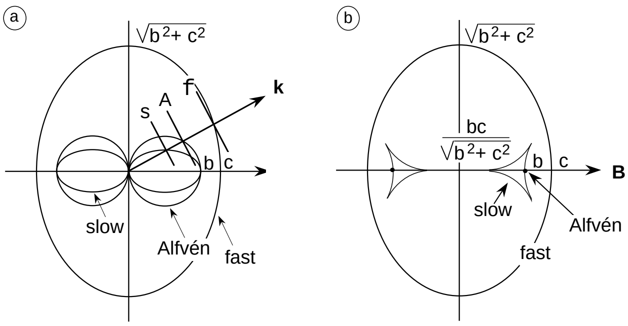

Figure 1: Schematic representation of a. phase diagram, b. group diagram of the three MHD waves.

5) In conclusion: The analysis of MHD waves in homogeneous plasmas shows that the spectral structure is determined by the three waves, that the Alfvén waves play a central role, and that the vanishing of the Alfvén wave frequency, described by the $\mathbf{B}\cdot\nabla$ operator, marks marginal stability. The question is how inhomogeneity in one (cylinder) and two (torus) dimensions modifies this picture and opens the door for instabilities.

2 Spectral theory

Two view points

How does one know whether a dynamical system is stable or not? Consider the well-known example of a ball at rest at the bottom of a trough or on the top of a hill (Fig. 2). There is a position where the potential energy W due to gravity has an extremum W_0. Displacing the ball slightly to a neighbouring position results in either a higher or a lower potential energy W_1. This corresponds to a *stable* system in the first case and an *unstable* system in the second.

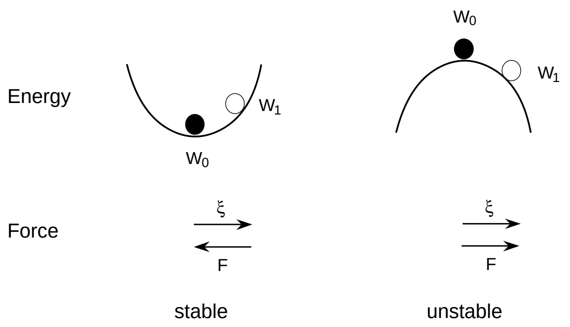

Figure 2: Two points of view: energy and force.

This simple example illustrates the general theoretical approach to linear stability, where the study of the original non-linear equations is simplified by means of a split in equilibrium and perturbations. Such a study may be conducted by means of two broad classes of methods, viz. by using variational principles involving *quadratic forms* (like the energy) or by solving *the differential equations* themselves. These methods are just a generalisation of the two intuitive approaches illustrated in Fig. 2. The upper part illustrates the investigation of stability by itself by means of the so-called *energy principle*, i.e. a study of the sign of the potential energy W of the perturbations ($W > 0$: stable, $W < 0$: unstable, $W = 0$: marginally stable). The full dynamics of the system may be obtained from a variational principle which not only involves the potential energy but also the kinetic energy of the perturbations. The more usual approach is the solution of differential equations, in particular the equation of motion, which involves a study of the *forces* acting on the system. With respect to stability, this method is illustrated in the bottom part of Fig. 2. If a displacement ξ creates a force F in the opposite direction, the state of equilibrium tends to be restored and the system is stable. On the other hand, if the resulting force is in the same direction as the displacement, the motion will be away from equilibrium and the system is unstable.

These intuitive notions on displacements, forces, and energies may be generalised to continuous media, in particular magneto-fluids. This leads to the *two alternative representations* by means of the equation of motion, involving the plasma displacement vector field $\boldsymbol{\xi}(\mathbf{r}, t)$ and the linear force operator $\mathbf{F}(\boldsymbol{\xi})$ acting on that field, on the one hand, and by means of variational quadratic forms, involving the potential energy functional $W[\boldsymbol{\xi}]$ and the kinetic energy functional $K[\boldsymbol{\xi}]$, on the other hand.

It is of some interest for our subject to note that the same alternative of a description by means of differential equations or by quadratic forms goes under the names of the Schrödinger and Heisenberg pictures in quantum mechanics. Let us see how it works out in MHD.

Force operator formalism

1) Starting point is *the nonlinear ideal MHD equations*, where we will suppress the factor μ_0 from now on:

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p + \mathbf{j} \times \mathbf{B}\,, \qquad \mathbf{j} = \nabla \times \mathbf{B}\,, \tag{1}$$

$$\frac{\partial p}{\partial t} = -\mathbf{v} \cdot \nabla p - \gamma p \nabla \cdot \mathbf{v}\,, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \tag{3}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})\,. \tag{4}$$

For simplicity, let the plasma be enclosed by a rigid wall. Disregarding trivial boundary conditions like regularity and periodicity, the non-trivial boundary conditions are the ones pertaining to the normal direction at the wall:

$$\mathbf{n} \cdot \mathbf{v} = 0\,, \qquad \mathbf{n} \cdot \mathbf{B} = 0 \qquad \text{(at the wall)}\,. \tag{5}$$

2) *Linearise* about a static equilibrium:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1\,, \tag{6}$$

$$\frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}_1\,, \tag{7}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)\,, \tag{8}$$

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}_1)\,, \tag{9}$$

where $p_0, \mathbf{B}_0, \mathbf{j}_0$ should satisfy the equilibrium equations (Eqs. (1)–(3) of the Equilibrium lecture).

3) Introducing *the Lagrangian displacement vector field* $\boldsymbol{\xi}$ (Fig. 3), related to the velocity perturbation

$$\mathbf{v}_1 = \frac{\partial \boldsymbol{\xi}}{\partial t}\,, \tag{10}$$

Eqs. (7)–(8) for the perturbations p_1, $\mathbf{B}_1$, ρ_1 can be integrated in time:

$$p_1 = -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi}\,, \tag{11}$$

$$\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)\,, \tag{12}$$

$$\rho_1 = -\nabla \cdot (\rho_0 \boldsymbol{\xi})\,. \tag{13}$$

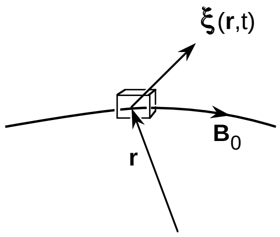

Figure 3: Displacement vector field.

Substitution back into the equation of motion (6) gives the *force operator* formulation [2]:

$$\mathbf{F}(\boldsymbol{\xi}) \equiv -\nabla p_1 - \mathbf{B} \times (\nabla \times \mathbf{B}_1) + (\nabla \times \mathbf{B}) \times \mathbf{B}_1 = \rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} , \qquad (14)$$

where we have dropped the subscript 0 on the equilibrium quantities. Note that ρ_1 does not enter the force operator. (This is different in astrophysical plasmas when gravity plays a role.)

Only one boundary condition remains, viz.

$$\mathbf{n} \cdot \boldsymbol{\xi} = 0 \qquad \text{(at the wall)} . \qquad (15)$$

This surprising fact (in view of the occurrence of three coupled second order differential equations) is associated with the strong anisotropy of the plasma perturbations with respect to the magnetic surfaces: the system (14) is of sixth order along the magnetic surfaces, but only of second order across.

4) Since the equilibrium quantities appearing in Eq. (14) do not depend on time we may consider solutions in the form of *normal modes*:

$$\boldsymbol{\xi}(\mathbf{r}, t) = \hat{\boldsymbol{\xi}}(\mathbf{r}) e^{-i\omega t} . \qquad (16)$$

This transforms Eq. (14) into an eigenvalue problem:

$$\mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho\omega^2 \hat{\boldsymbol{\xi}} , \qquad (17)$$

with the linear operator $\mathbf{F}$ (or rather $\rho^{-1}\mathbf{F}$) having eigenvalues $-\omega^2$. Except for discrete eigenvalues, ideal MHD also allows for continuous (or 'improper') eigenvalues. The collection of these two kinds of eigenvalues is called *the spectrum of ideal MHD*. The most important property of the operator $\rho^{-1}\mathbf{F}$ is that it is *self-adjoint* [2] so that *the eigenvalues* ω^2 *are real*. (See point 9).)

5) The reality of ω^2 implies that two quite different classes of solutions occur, viz. *stable waves* with harmonic time-dependence for $\omega^2 > 0$, and *instabilities* with exponentially explosive time-dependence for $\omega^2 < 0$. For eigenmodes, we now see the connection with the bottom part of the pictures of Fig. 2 on the relationship between displacements and forces: According to Eq. (17), $\mathbf{F} \sim -\hat{\boldsymbol{\xi}}$ for stable waves ($\omega^2 > 0$) and $\mathbf{F} \sim \hat{\boldsymbol{\xi}}$ for instabilities ($\omega^2 < 0$). For general motions, consisting of a superposition of eigenmodes, such a simple relationship does not hold. In that case, we have to turn to the alternative view point of quadratic forms, corresponding to the upper part of Fig. 2, and study the sign of the potential energy. (See point 12).)

6) In *dissipative* (e.g. resistive) MHD different eigenmodes are possible. In particular, since ω^2 need not be real, complex values of ω may occur. This may give rise to stable, but damped, modes if $\mathrm{Im}(\omega) < 0$ and so-called 'overstable' modes if $\mathrm{Im}(\omega) > 0$. These additional possibilities are associated with the fact that the waves are non-conservative in dissipative MHD: energy may be dissipated or accumulated.

7) Clearly, ideal, conservative MHD presents a significant simplification for stability problems. Since ω^2 is real, the transition from stability to instability occurs only through the value $\omega^2 = 0$, i.e. through *marginal stability*. Consequently, to study the problem of stability one could study the marginal equation of motion

$$\mathbf{F}(\hat{\boldsymbol{\xi}}) = 0 , \qquad (18)$$

subject to the b.c. (15). In general, this equation has no solution because Eq. (17) is an eigenvalue problem and $\omega^2 = 0$ does not have to be an eigenvalue.

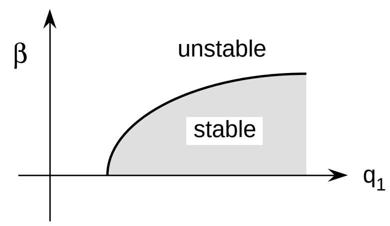

Figure 4: Schematic stability diagram for tokamaks.

In order to get genuine solutions one should arrange the equilibrium parameters such that this becomes true. E.g., a typical tokamak stability study would involve the variation of global equilibrium parameters like the value of $\beta \equiv 2\mu_0 p/B^2$ and $q_1 \sim 1/I_p$, while keeping other variables fixed. For a particular value of the plasma current I_p one would push the value of β until the marginal equation of motion (18) would be satisfied, subject to the boundary condition (15). In this manner, one would calculate one value β_{crit} where marginal stability is obtained. By varying the value of q_1 one would trace out marginal stability curves in the β–q_1 diagram, as schematically shown in Fig. 4. Physical arguments usually indicate on which side of the curve the stable states are to be found (e.g., in Fig.4, this would be on the low β side). This is the most general, though not the easiest, method of studying stability problems.

Quadratic forms and stability

8) To arrive at a simpler method for studying stability, reflecting the intuitive notions depicted in the upper part of Fig. 2, we now turn to the quadratic forms that may be defined for linearised MHD. First of all, define the inner product of different plasma displacement fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ with the plasma equilibrium density $\rho(\mathbf{r})$ acting as a kind of weight function:

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle \equiv \tfrac{1}{2} \int \rho\, \boldsymbol{\xi}^* \cdot \boldsymbol{\eta}\, dV \,. \tag{19}$$

This also permits to define *the norm* of vectors $\boldsymbol{\xi}$:

$$I \equiv \|\boldsymbol{\xi}\|^2 \equiv \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = \tfrac{1}{2} \int \rho\, \boldsymbol{\xi}^* \cdot \boldsymbol{\xi}\, dV \,. \tag{20}$$

For functions with a finite norm, $\|\boldsymbol{\xi}\| < \infty$, a linear function space is obtained, viz. *Hilbert space*.

The physical background for considering vector fields $\boldsymbol{\xi}(\mathbf{r}, t)$ with a finite norm is the expression for *the kinetic energy* of the plasma:

$$K \equiv \tfrac{1}{2} \int \rho \mathbf{v}^2\, dV \approx \tfrac{1}{2} \int \rho |\dot{\boldsymbol{\xi}}|^2\, dV = \langle \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}} \rangle \,. \tag{21}$$

As far as the spatial dependence is concerned, $\dot{\boldsymbol{\xi}}(\mathbf{r}, t)$ belongs to the same class of functions as $\boldsymbol{\xi}(\mathbf{r}, t)$ (t is simply a parameter). Hence, a bounded norm implies finite kinetic energy of the perturbations. Because the total energy is conserved, this also implies finite potential energy: a very reasonable assumption, justifying the use of Hilbert space as a theoretical tool.

9) We can now formulate the central property of linearised ideal MHD: *the force operator* $\mathbf{F}$, or rather $\rho^{-1}\mathbf{F}$, *is a self-adjoint linear operator in the Hilbert space of plasma displacement vectors*:

$$\int \boldsymbol{\eta}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV = \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\eta})\, dV \,. \tag{22}$$

This provides linear MHD with a mathematical foundation that is the same as that of quantum mechanics, viz. linear operators in Hilbert space! Hence, many analogies are to be expected between MHD and quantum mechanical spectral theory.

10) Self-adjointness expresses the basic property of *energy conservation*: $W + K = \text{const}$, where W is the potential energy and K is the kinetic energy of the perturbations. We have defined the kinetic energy in Eq. (21). We could find the corresponding expression for the potential energy of the perturbations by expanding $\int [\, p/(\gamma - 1) + \tfrac{1}{2} B^2 \,]\, dV$ order by order. However, we would have to develop to second order so that it is easier to exploit energy conservation directly, in the following manner. Take the inner product of $\dot{\boldsymbol{\xi}}^*$ with the equation of motion (14) and integrate over the plasma volume:

$$\int \dot{\boldsymbol{\xi}}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV = \int \rho\, \dot{\boldsymbol{\xi}}^* \cdot \ddot{\boldsymbol{\xi}}\, dV = \dot{K} \,. \tag{23}$$

Exploiting energy conservation and the self-adjointness of $\mathbf{F}$ one finds

$$\dot{W} = -\dot{K} = \frac{d}{dt} \Big[-\tfrac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV \Big] \,,$$

so that

$$W = -\tfrac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV \,, \tag{24}$$

which is the expression for *the potential energy* we were looking for. The result is plausible: The potential energy W is the amount of energy obtained by the plasma through the work of the force $\mathbf{F}$ against the displacement $\boldsymbol{\xi}$. The factor $\tfrac{1}{2}$ represents the averaging involved as the work builds up when the plasma is displaced from 0 to its actual value $\boldsymbol{\xi}$.

The expression (24) for W may be transformed into the more useful form

$$W = \tfrac{1}{2} \int \Big[\gamma p\, |\nabla \cdot \boldsymbol{\xi}|^2 + |\mathbf{B}_1|^2 + (\boldsymbol{\xi}^* \cdot \nabla p)\, \nabla \cdot \boldsymbol{\xi} + \mathbf{j} \cdot \boldsymbol{\xi}^* \times \mathbf{B}_1 \Big]\, dV \,. \tag{25}$$

The terms represent, successively, the acoustic and the magnetic energy, which are positive definite so that *homogeneous plasmas are always stable*, and the pressure gradient and the current driven energy terms, which may have either sign so that *inhomogeneous plasmas may be unstable*.

Having obtained the expressions for the kinetic and potential energy, one may define a Lagrangian and derive the dynamics from *Hamilton's principle*. This is completely equivalent with the equation of motion (14).

11) From the equation of motion (14) we obtained the spectral equation (17) by considering normal modes (16). It is useful to also consider the quadratic forms for normal modes. These are obtained by inserting (16) into the quadratic forms (21) for K and (24) for W:

$$\omega^2 = \frac{W[\hat{\boldsymbol{\xi}}]}{I[\hat{\boldsymbol{\xi}}]} \qquad \text{for normal modes} \,. \tag{26}$$

Note that this expression is just a conclusion a posteriori, after the normal modes have been obtained. A recipe for finding the eigenvalues ω^2 and the eigenfunctions $\hat{\boldsymbol{\xi}}$ is only obtained by considering the right hand side of Eq. (26) to be a variational expression in $\hat{\boldsymbol{\xi}}$.

Variational principle: The eigenfunctions of the operator $\mathbf{F}$ make the functional

$$\Omega^2[\hat{\boldsymbol{\xi}}] \equiv \frac{W[\hat{\boldsymbol{\xi}}]}{I[\hat{\boldsymbol{\xi}}]} \,, \tag{27}$$

Differential equations ('Schrödinger')	**Quadratic forms** ('Heisenberg')	
$\mathbf{F}(\boldsymbol{\xi}) = \rho \dfrac{\partial^2 \boldsymbol{\xi}}{\partial t^2}$ *(Eq. of motion)*	$\delta \displaystyle\int_{t_1}^{t_2} dt \,\big(K[\dot{\boldsymbol{\xi}}] - W[\boldsymbol{\xi}]\,\big) = 0$ *(Hamilton's principle)*	$\Rightarrow$ Full dynamics: $\boldsymbol{\xi}(\mathbf{r}, t)$
$\mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho\omega^2 \hat{\boldsymbol{\xi}}$ *(Eigenvalue problem)*	$\delta \dfrac{W[\hat{\boldsymbol{\xi}}]}{I[\hat{\boldsymbol{\xi}}]} = 0$ *(Rayleigh's principle)*	$\Rightarrow$ Spectrum $\{\omega^2\}$ + eigenfcts. $\{\hat{\boldsymbol{\xi}}(\mathbf{r})\}$
$\mathbf{F}(\hat{\boldsymbol{\xi}}_0) = 0$ *(Marg. eq. of motion)*	$W[\hat{\boldsymbol{\xi}}_0] \gtrless 0$ *(Energy principle)*	$\Rightarrow$ Stability (y/n) $\big($+ trial fct. $\hat{\boldsymbol{\xi}}_0(\mathbf{r})\big)$

Figure 5: The two 'pictures' of ideal MHD.

stationary; the eigenvalues are the stationary values of Ω^2. In this form the variational principle and the eigenvalue equation (17) are equivalent. The variational principle provides the basis for effective numerical methods.

12) We now have a formulation that is one step more useful than the force operator equation. Since $I[\hat{\boldsymbol{\xi}}] \geq 0$, we may insert *trial functions* in the expression (27). If $W[\hat{\boldsymbol{\xi}}] > 0$ for all possible trial functions $\hat{\boldsymbol{\xi}}$, we may conclude that no eigenvalues $\omega^2 < 0$ exist and the system is *stable*. On the other hand, if one can find a single $\hat{\boldsymbol{\xi}}$ for which $W[\hat{\boldsymbol{\xi}}] < 0$, at least one negative eigenvalue ω^2 exists and the system is *unstable*. In many cases this is a much quicker way to establish instability of a particular configuration than to study the marginal equation of motion (18). The procedure is summarised in the following powerful statement.

Energy principle: An equilibrium is stable if (sufficient) and only if (necessary)

$$W[\hat{\boldsymbol{\xi}}] > 0 \qquad (28)$$

for all displacements $\hat{\boldsymbol{\xi}}(\mathbf{r})$ that are bound in norm and satisfy the boundary conditions.

We can now complete the discussion of Fig. 2 on stability when the perturbation is not an eigenfunction so that one can not determine whether $\boldsymbol{\xi}$ and $\mathbf{F}$ have the same or opposite sign. Clearly, in that case the energy point of view provides a clear-cut answer since the relative sign of $\boldsymbol{\xi}$ and $\mathbf{F}$ is accounted for in an integrated sense given by the definition (24) of the potential energy so that one can determine whether $W > 0$ or < 0.

13) Specialising to stability, the variational approach offers three different methods for determining stability:

a. By physical intuition one may *guess* a trial function $\hat{\boldsymbol{\xi}}(\mathbf{r})$ that picks up the unstable part of the potential energy, so that $W[\hat{\boldsymbol{\xi}}] < 0$. This provides a direct demonstration of the instability of a certain system: *necessary stability criteria* are obtained this way;
b. More systematically, one may consider *a complete set of trial functions that are normalised in any convenient way*, e.g. by fixing only the part referring to the perturbation perpendicular to the magnetic surfaces, $\int \rho\,(\mathbf{n}\cdot\boldsymbol{\xi})^2\, dV$ (i.e., not I itself): *necessary + sufficient criteria* for stability are obtained;
c. Finally, by considering *a complete set of trial functions that are properly normalised with the norm* $I[\hat{\boldsymbol{\xi}}]$ *the complete spectrum* of eigenvalues $\{\omega^2\}$ is obtained.

Comparing the variational with the differential equations approach, notice that:
Method a. has no counterpart in the equation of motion approach;
Method b. is equivalent to solving the marginal equation of motion $\mathbf{F}(\hat{\boldsymbol{\xi}}) = 0$;
Method c. is equivalent to solving the spectral equation $\mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho\omega^2\hat{\boldsymbol{\xi}}$.

In conclusion, *the analogy with quantum mechanics is complete*: We have obtained two pictures (Fig. 5), corresponding to the Schrödinger picture of wave mechanics (with a description in terms of the wave equation $H\psi = E\psi$) and the Heisenberg picture of matrix mechanics (with a description in terms of the representative matrix elements $\langle n|H|m\rangle$ of the Hamiltonian), viz. that of *the equation of motion* and that of *the variational principle* exploiting the quadratic forms of the potential and kinetic energy. The analogy is mathematical, not physical. The physical systems are completely different. E.g., the transition from bound to free states in the spectrum of quantum mechanics corresponds to the transition from stable to unstable modes in magnetohydrodynamics. Also, remember that the force operator is not a Hamiltonian.

3 Waves and instabilities in cylinder geometry

The first application of MHD spectral theory is the stability analysis of diffuse plasmas in cylinder geometry. The second important application is, of course, toroidal confinement systems, which is the subject of the next lecture. Note that this way of presenting things is at variance with the historical development: the specific stability results came first and spectral theory was developed much later. E.g., the beautiful paper by Newcomb [5], which logically incorporated the Suydam criterion [4], was written from the point of view of stability alone (i.e., $\omega^2 = 0$ in the equation of motion). The earlier paper by Hain and Lüst [3], which gave the more general equation for the waves ($\omega^2 \neq 0$), was not exploited until later in the seventies when it was realised that marginal stability is just a special case of the more general spectral problem of wave propagation in inhomogeneous media. Here, we will reverse the order and present the Hain-Lüst equation first, then indicate the spectral implications, and finally treat the stability of diffuse pinches and straight tokamaks as special applications.

Hain-Lüst equation

1) Because of cylinder symmetry we study *normal mode solutions* of the form

$$\boldsymbol{\xi}(r,\theta,z,t) = \big(\xi_r(r), \xi_\theta(r), \xi_z(r)\big)\, e^{i(m\theta + kz - \omega t)} \,. \quad (29)$$

For these separate modes the equation of motion (17) may be reduced to an ordinary second order differential equation in terms of the component $\xi_r(r)$.

2) We exploit *a projection based on the magnetic surfaces and field lines*:

$$\mathbf{n} \equiv \mathbf{e}_r \,, \quad \mathbf{e}_\perp \equiv (\mathbf{B}/B) \times \mathbf{n} \,, \quad \mathbf{e}_\parallel \equiv \mathbf{B}/B \,. \quad (30)$$

In this projection the gradient of a perturbed scalar quantity may be written as

$$\begin{aligned} \nabla &= \mathbf{n}\,\frac{\partial}{\partial r} + \mathbf{e}_\perp\, iG/B + \mathbf{e}_\parallel\, iF/B \,, \\ G &\equiv -i\,\mathbf{B}\times\mathbf{n}\cdot\nabla = mB_z/r - kB_\theta \,, \\ F &\equiv -i\,\mathbf{B}\cdot\nabla \qquad = mB_\theta/r + kB_z \,, \end{aligned} \quad (31)$$

so that the tangential gradient operators F and G are algebraic. The components of $\boldsymbol{\xi}$ are denoted as

$$\begin{aligned} X &\equiv r\,\mathbf{n}\cdot\boldsymbol{\xi} = r\,\xi_r \,, \\ Y &\equiv ir\,\mathbf{e}_\perp\cdot\boldsymbol{\xi} = ir(B_z\xi_\theta - B_\theta\xi_z)/B \,, \\ Z &\equiv ir\,\mathbf{e}_\parallel\cdot\boldsymbol{\xi} \ = ir(B_\theta\xi_\theta + B_z\xi_z)/B \,. \end{aligned} \quad (32)$$

3) This leads to the following *the spectral problem*:

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = -\rho\omega^2 \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \,, \quad (33)$$

where the matrix elements involve the radial gradient operator:

$$\begin{aligned} \mathcal{A}_{11} &\equiv r\frac{d}{dr}\frac{1}{r}\,(\gamma p + B^2)\,\frac{d}{dr} - F^2 - r\Big(\frac{B_\theta^2}{r^2}\Big)' \,, \\ \mathcal{A}_{12} &\equiv r\frac{d}{dr}\frac{1}{r}\frac{G}{B}\,(\gamma p + B^2) - 2k\frac{B_\theta B}{r} \,, \\ \mathcal{A}_{13} &\equiv r\frac{d}{dr}\frac{1}{r}\frac{F}{B}\,\gamma p \,, \\ \mathcal{A}_{21} &\equiv -\frac{G}{B}\,(\gamma p + B^2)\,\frac{d}{dr} - 2k\frac{B_\theta B}{r} \,, \\ \mathcal{A}_{22} &\equiv -\frac{G^2}{B^2}\,(\gamma p + B^2) - F^2 \,, \\ \mathcal{A}_{23} &\equiv -\frac{GF}{B^2}\,\gamma p \,, \\ \mathcal{A}_{31} &\equiv -\frac{F}{B}\,\gamma p\,\frac{d}{dr} \,, \\ \mathcal{A}_{32} &\equiv -\frac{FG}{B^2}\,\gamma p \,, \\ \mathcal{A}_{33} &\equiv -\frac{F^2}{B^2}\,\gamma p \,. \end{aligned} \quad (34)$$

4) *The tangential components* can be expressed in terms of the radial variable X :

$$\begin{aligned} Y &= \frac{G\,[\,(\gamma p + B^2)\rho\omega^2 - \gamma p F^2\,]\, rX'}{rBD} \\ &\quad + \frac{2kB_\theta(B^2\rho\omega^2 - \gamma p F^2)\,X}{rBD} \,, \\ Z &= \frac{\gamma p F\,[\,(\rho\omega^2 - F^2)\, rX' + 2kB_\theta G\,X\,]}{rBD} \,, \end{aligned} \quad (35)$$

where the denominator

$$D \equiv \rho^2\,[\,\omega^2 - \omega_I^2(r)\,]\,[\,\omega^2 - \omega_{II}^2(r)\,]$$

involves the two 'separator' frequencies

$$\begin{aligned} \omega_{I,II}^2 &\equiv \tfrac{1}{2}\rho^{-1}(m^2/r^2 + k^2)(\gamma p + B^2) \\ &\quad \times \left[1 \pm \sqrt{1 - \frac{4\gamma p F^2}{(m^2/r^2 + k^2)(\gamma p + B^2)^2}}\,\right] . \end{aligned} \quad (36)$$

5) One can then *reduce the system to a single second order differential equation* by substituting these expressions into the first component of Eq. (33). This gives the *Hain-Lüst equation*:

$$\Big[\frac{N}{rD}X'\Big]' + \Big[\frac{1}{r}(\rho\omega^2 - F^2) - \Big(\frac{B_\theta^2}{r^2}\Big)' - \frac{4k^2B_\theta^2}{r^3D}(B^2\rho\omega^2 - \gamma p F^2) + \Big\{\frac{2kB_\theta G}{r^2D}\big((\gamma p + B^2)\rho\omega^2 - \gamma p F^2\big)\Big\}'\Big]X = 0\,, \quad (37)$$

where the numerator

$$N \equiv \rho^2(\gamma p + B^2)\,[\,\omega^2 - \omega_A^2(r)\,]\,[\,\omega^2 - \omega_S^2(r)\,]$$

involves the Alfvén and slow continuum frequencies

$$\omega_A^2 \equiv \rho^{-1}F^2\,, \qquad \omega_S^2 \equiv \rho^{-1}\frac{\gamma p}{\gamma p + B^2}F^2\,. \quad (38)$$

6) Appropriate boundary conditions for Eq. (37) are

$$X(0) = X(a) = 0 \quad (39)$$

if the wall is at the plasma ($b = a$). We recall the discussion of the boundary condition (15). It is now clear that no more than two boundary conditions are needed since Eq. (37) for X is a second order ODE and the expressions (35) for Y and Z are algebraic. In toroidal geometry the tangential gradient operators are no longer algebraic, but the formal reduction corresponding to Eq. (35) does not raise the order of the system with respect to normal derivatives.

Spectral implications

The above derivation involves a number of peculiar quantities, in particular the expressions for N and D, which are indicative of the different aspects of the underlying spectral structure. We here state them without proof. (The interested reader is referred to the literature quoted and to Refs. [17] and [21] for MHD spectral theory. The analysis given is an elaboration of Refs. [8] where the equations for the diffuse cylinder and the plane gravitating slab were derived side by side.)

7) Clearly, the three-fold spectrum of MHD waves, reflected in the Friedrich's diagram of Fig. 1, is substantially modified by the introduction of inhomogeneities in the plasma equilibrium. This is evident from the complicated structure of the Hain-Lüst equation (37). This equation contains two quite different contributions, viz. the factor N/D in front of the highest derivative and terms which depend on the field line curvature B_θ. The latter ones are responsible for possible instabilities, but they do not affect the highest derivative terms. Vice versa, the quantities $N(r;\omega^2)$ and $D(r;\omega^2)$ do not depend on the presence of field line curvature. In a sense, they can be considered as representing the state of affairs inside a particular magnetic surface. Hence, we can anticipate some properties of inhomogeneous plasmas by studying the dispersion equation $\omega^2 = \omega^2(k_x, k_y, k_z)$ of homogeneous plasmas in the limit of large wave numbers k_x, so as to mimic wave motion localised in the direction of inhomogeneity, represented by x. In the limit $k_x \to \infty$ one then finds the following features:

a. The eigenvalues ω_a^2 of the Alfvén subspectrum are *infinitely degenerate* since $\omega_A^2 \equiv \omega_a^2$ is independent of k_x;

b. The slow wave subspectrum monotonically decreases and it has an *accumulation point* at $\omega_S^2 \equiv \lim_{k_x\to\infty}\omega_s^2$;

c. The fast subspectrum is monotonically increasing so that $\omega_F^2 \equiv \lim_{k_x\to\infty}\omega_f^2 = \infty$ is a formal *cluster point* of the fast wave point spectrum.

This asymptotic structure also presents the essential spectrum of the inhomogeneous case, as we will see.

The two special values of ω^2 denoted by ω_I^2 and ω_{II}^2 also already appear in the homogeneous analysis. They are related to the opposite extreme, viz., to the limit $k_x \to 0$ when *the fast and slow modes change character from propagating to evanescent*. Their values are given by $\omega_{I,II}^2 \equiv \omega_{s,f}^2(k_x = 0)$.

8) In the inhomogeneous case, the infinite degeneracy of the Alfvén eigenvalues is lifted by the appearance of a continuum of improper Alfvén modes and the slow cluster point is spread out in a continuum of improper slow modes. Hence, the local Alfvén frequencies $\{\omega_A^2\}$ and slow frequencies $\{\omega_S^2\}$ on the interval $0 \le r \le a$ constitute *continuous spectra* [6], [9].

The improper Alfvén eigenfunctions exhibit singular behaviour at the position $r = r_A(\omega^2)$, where $r_A = r_A(\omega^2)$ is the inversion of the Alfvén continuum profile $\omega_A^2 = \omega_A^2(r)$:

$$X_A = u(r)\big[\ln|r - r_A| + \lambda\,H(r - r_A)\big] + v(r)\,, \quad (40)$$

where u and v are regular functions and the constant λ is arbitrary. However, this is just the secondary behaviour. The dominant non-square integrable part resides in the perpendicular tangential component Y for the Alfvén waves:

$$Y_A \approx \mathcal{P}\frac{1}{r - r_A(\omega^2)} + \lambda\,\delta(r - r_A(\omega^2))\,. \quad (41)$$

Similarly, for the improper slow eigenfunctions, with the subscript A replaced by S and the parallel tangential component Z_S being the dominant contribution.

9) Similarly, the local values of $\omega_{I,II}^2$ also spread out when inhomogeneity is introduced. However, the range of frequencies $\{\omega_I^2\}$ and $\{\omega_{II}^2\}$ for $0 \le r \le a$ *do not constitute continuous spectra* [13]. They just mark regions

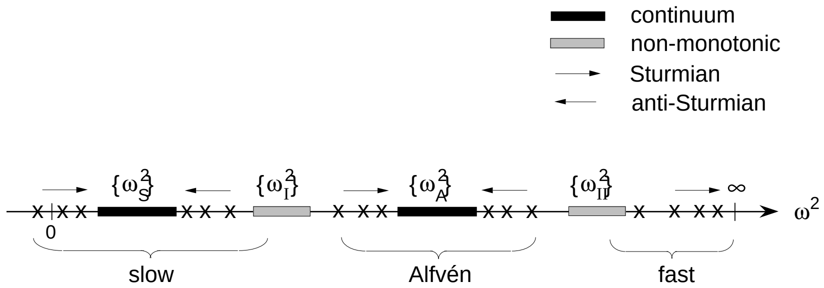

Figure 6: Schematic structure of the spectrum of an inhomogeneous plasma.

of the spectrum where the different kinds of discrete modes exhibit *non-monotonicity* [11].

10) Implications of the continuous spectrum are *decay* of Alfvén waves by phase mixing [10] and *heating* by resonant absorption [12], [14], with the potential of providing a theory on the heating of the coronae of x-ray emitting stars [23], [24]. (Because of our present subject, we have restricted references to the growing astrophysical literature to these recent ones, where also further references may to be found.)

11) The final structure of the spectrum for inhomogeneous plasmas becomes rather complicated because, in general, the continua $\{\omega^2_{A,S}\}$ cover a substantial part, if not all, of the positive axis $\omega^2 \geq 0$ and, moreover, they may overlap or fold over onto themselves. A schematic presentation can be given for the case of weak inhomogeneity: Fig. 6 (taken from Ref. [15]). Except for the formation of the continua, inhomogeneity results in a new subspectrum of discrete slow modes, clustering from below at the slow continuum, and two new subspectra of discrete Alfvén modes, clustering both from below and from above at the Alfvén continuum, whereas discrete fast modes remains characterised by clustering at ∞. Thus, the three-fold spectrum of MHD waves in homogenous plasmas is split into a *five-fold spectrum in inhomogeneous plasmas*. Monotonicity with respect to the number of nodes of the radial component X of the eigenfunction of the discrete modes can be proved for frequencies outside the two continua $\{\omega^2_{A,S}\}$ and the separator frequencies $\{\omega^2_{I,II}\}$: *the oscillation theorem* [11]. In those ranges, the discrete subspectra exhibit *Sturmian* (eigenvalues increase with number of nodes of the eigenfunction) or *anti-Sturmian* (opposite) behaviour.

12) With respect to stability, spectral theory offers two important insights:
a. The oscillation theorem, which is valid over the entire unstable range $\omega^2 < 0$, provides the connection between marginal stability results (Newcomb's theory) and actual growth rates of instabilities.
b. Clustering of discrete modes at the tips of the continua occurs on the stable side of the spectrum when $\{\omega^2_A\}$ or $\{\omega^2_S\}$ has an extremum and certain additional conditions are met [18]. Minima of the two continua always occur at marginal stability when $\mathbf{B}\cdot\nabla = iF = 0$ somewhere. Thus, local stability conditions (Suydam's criterion in a cylinder, Mercier's criterion, and also certain aspects of ballooning theory in a torus) are just special cases of the more general phenomenon of clustering. (Numerical example of unstable clusterpoint: see [22].)

Newcomb's stability analysis

13) An example of the use of the marginal equation (18), obtained by minimising the energy principle, is the stability analysis of Newcomb [5] and Suydam[4]. Starting point is the one-dimensional energy principle,

$$W[\xi_0] = \pi L \int_0^a (f_0\, {\xi_0'}^2 + g_0\, \xi_0^2)\, dr\,, \tag{42}$$

where L is the length of the plasma column. The minimising Euler-Lagrange equation takes the form

$$(f_0\, \xi_0')' - g_0\, \xi_0 = 0\,, \tag{43}$$

where

$$\begin{aligned} f_0 &= \frac{r^3F^2}{m^2+k^2r^2}\,,\\ g_0 &= \frac{2k^2r^2}{m^2+k^2r^2}\,p' + \frac{m^2+k^2r^2-1}{m^2+k^2r^2}\,rF^2\\ &\quad -\frac{2k^2r^3(mB_\theta/r - kB_z)}{(m^2+k^2r^2)^2}\,F\,. \end{aligned} \tag{44}$$

This equation may be transformed to the marginal Hain-Lüst equation (37) with $\omega^2 = 0$:

$$\Big[\frac{rF^2}{m^2+k^2r^2}\,X_0'\Big]' - \Big[\frac{1}{r}F^2 + \Big(\frac{B_\theta^2}{r^2}\Big)' - \frac{4k^2B_\theta^2}{r(m^2+k^2r^2)} + \Big(\frac{2kB_\theta G}{m^2+k^2r^2}\Big)'\Big]\,X_0 = 0\,. \tag{45}$$

The equations (44) and (45) are equivalent, but the derivation from the Hain-Lüst equation has the advantage that the connection with genuine eigenvalues is transparent: Whenever the marginal equation of motion (45) permits oscillatory solutions, the oscillation theorem guarantees that these zeros will move away from each other monotonically as the parameter ω^2 is decreased until they hit the boundaries so that a genuine global unstable mode is obtained.

14) The other major obstacle cleared by Newcomb's analysis, is the elucidation of the singularities $F = 0$ and the meaning of Suydam's criterion. This involves the investigation of the solutions to the Euler equation (43) in the neighbourhood of a singularity $F = 0$. Close to such a singularity Eq. (43) reduces to

$$(s^2\xi_0')' - \alpha\,\xi_0 = 0\,, \quad \alpha \equiv 2q^2p'/\big(rB_z^2q'^2\big)\,, \qquad (46)$$

having solutions s^{n_1} and s^{n_2} , where

$$n_{1,2} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4\alpha}\,. \qquad (47)$$

Depending on whether $1+4\alpha$ is positive or negative the indices are real or complex. When the indices are complex, the real solutions to the Euler equation are oscillatory,

$$\xi_0 \sim s^{-1/2}\sin\big(\tfrac{1}{2}\sqrt{-1-4\alpha}\,\ln s + \mathrm{const}\big)\,, \qquad (48)$$

and, hence, the system is unstable. For $s \to 0$ these solutions oscillate infinitely rapidly, and their amplitude also blows up (Fig. 7a). It is clear that this behaviour is indicative of a spectral clusterpoint: By decreasing ω^2 the zeros will disappear one by one, so that an infinite sequence of unstable modes is obtained where the most global mode is the fastest growing (Fig. 7b).

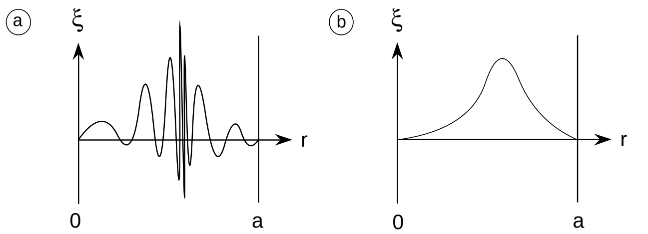

Figure 7: Violation of Suydam's criterion: a. marginal mode ($\omega^2 = 0, n = \infty$), b. associated global mode ($\omega^2 < 0, n = 0$).

The condition $1+4\alpha > 0$ which is necessary for the absence of the oscillatory solutions (48) was derived first by Suydam and is, therefore, known as *Suydam's criterion*:

$$p' + \tfrac{1}{8}rB_z^2\Big(\frac{q'}{q}\Big)^2 > 0\,. \qquad (49)$$

It is the simplest example of the competition between the driving force of interchange instabilities (pressure gradient) and stabilising magnetic shear.

Stability of 'straight' tokamaks

We finish with a description of that part of cylinder theory that may be considered as a first approximation to the stability of toroidal tokamaks, i.e. the leading order of the so-called *low-β tokamak approximation*. This implies imposing periodic boundary conditions to the perturbation of a cylinder of finite length. Pressure effects are neglected.

15) The plasma energy expression (42) is extended with the potential energy of the external vacuum magnetic field perturbations (inside a wall at $r = b$). Significant simplification occurs due to the tokamak ordering, i.e. $\epsilon \equiv a/R_0 \ll 1$, $q \approx aB_z/(R_0B_\theta) \sim 1$, $\beta \sim \epsilon^2$ (i.e. negligible), and $ka \approx na/R_0 \sim \epsilon \ll 1$ (long wavelength approximation). Expanding the potential energy order by order, $W = W^{(2)} + \epsilon^2 W^{(4)} + \cdots$, the leading order contribution turns out to be of second order, whereas the next (fourth) order requires toroidal theory since it would bring in pressure effects. Introducing a dimensionless radius $\bar r \equiv r/a$ (and dropping the bar),

$$\begin{aligned}\bar W &\equiv \frac{W^{(2)}}{\epsilon^2\cdot 2\pi^2a^2R_0B_0^2} \qquad (50)\\ &\approx \int_0^1 \Big(\frac{1}{q(r)} + \frac{n}{m}\Big)^2\big[\,r^2\xi'^2 + (m^2-1)\xi^2\,\big]\,rdr\\ &+\Big(\frac{1}{q_1^2} - \frac{n^2}{m^2}\Big)\xi_1^2 + |m|\Big(\frac{1}{q_1}+\frac{n}{m}\Big)^2\frac{1+(b/a)^{-2|m|}}{1-(b/a)^{-2|m|}}\xi_1^2.\end{aligned}$$

For given profile $q(r)$ and given wall position b/a, the problem is completely determined: for stability, we have to study the sign of $\bar W$ for all values of m and n.

16) From these expressions, it is immediately clear that *fixed-boundary* ($\xi_1 = 0$), *internal $|m| \geq 2$ modes are stable* since the integrand of the plasma integral (50) is positive definite. At worst, we can have marginal stability to leading order ($W^{(2)} = 0$) for the $|m| = 1$ modes when $q_0 < 1$. In that case a trial function can be chosen which is constant inside the $q = 1$ surface and vanishes outside. The jump at the latter surface can be permitted since it does not contribute to $W^{(2)}$. Toroidal modifications $W^{(4)}$ will determine whether the internal kink mode is stable or not. However, when $q_0 > 1$ there is no singularity and $W \approx W^{(2)} > 0$, so that *all internal kink modes are stable when $q_0 > 1$*.

The only modes left are the free-boundary, external kink modes. For $m = -1$, minimisation of the plasma integral is trivial: $\xi' = 0$ so that the perturbation is constant across the plasma (just a rigid helical motion of the plasma column). From the resulting expression for $\bar W$ it follows that *the $m = -1$ external kink modes are unstable for $a^2/b^2 < nq_1 < 1$, independent of the current profile*. To avoid these modes in tokamaks (with monotonic q profile) one should limit the toroidal current to a value such that the toroidal wavelength of these modes does not fit into the torus:

$$q_1 > 1 \qquad \textit{(Kruskal-Shafranov limit)}\,. \tag{51}$$

This is the ultimate limit on the current in tokamaks.

For $|m| \geq 2$ the minimising Euler equation is:

$$\Big[\Big(\frac{1}{q}+\frac{n}{m}\Big)^2 r^3\xi'\Big]' - (m^2-1)\Big(\frac{1}{q}+\frac{n}{m}\Big)^2 r\xi = 0\,. \tag{52}$$

Substituting the solution into Eq. (50) for $\bar{W}$ gives

$$\bar{W} = \frac{m+nq_1}{m^2q_1^2}\Big[(m+nq_1)\Big(\frac{\xi'}{\xi}\Big)_1 - m + nq_1 + |m|(m+nq_1)\,\frac{1+(b/a)^{-2|m|}}{1-(b/a)^{-2|m|}}\Big]\xi_1^2\,. \tag{53}$$

Eq. (52) can be solved explicitly for a flat current profile: $\xi \sim r^{|m|-1}$. Substitution in Eq. (53) shows that the system is unstable for all m in the respective ranges

$$|m| - 1 + (a^2/b^2)^{2|m|} \leq nq_1 \leq |m|\,. \tag{54}$$

For more realistic current distributions Shafranov [7] showed that *stability is obtained for sufficiently peaked current profiles*. The systematic investigation requires numerical integration of Eq. (52). For the resulting stability diagrams, see: Refs. [16], [20], and [19].

4 Conclusions

Alfvén wave dynamics occupies a central role in the description of plasma stability. This is due to the fact that the Alfvén waves are described by the parallel gradient operator, the action of which has to be minimised if instability is to occur, so that the most unstable perturbations are localised with respect to certain field lines and magnetic surfaces. This leads to all the ramifications of MHD spectral theory. The generalisations to toroidal geometry and resistive MHD will be described in the lecture on "Toroidal MHD instabilities".

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