for the particles to achieve this in a classical process with the help of their thermal energies is $T \approx 3 \cdot 10^9\,\mathrm{K}$. Fortunately the tunneling effect makes markedly lower temperatures possible. In a D-T reaction the quantum probability for penetrating the Coulomb wall is given by the *Gamow factor*

$$w \approx \exp\left(-34.4\sqrt{\mathrm{keV}/E_{\mathrm{kin}}}\right) . \tag{3}$$

For $E_{\mathrm{kin}} = 10$ keV we obtain a tunneling probability of $w \approx 1.9 \cdot 10^{-5}$, indicating that markedly lower temperatures than the classically required 3 billion Kelvin can lead to fusion.

According to our curve of binding energies the direct fusion of the ^{4}He-nucleus out of its four nucleons would still be more energetic than the D-T reaction because a total binding energy of 28 MeV would be released in this process, i.e. 7 MeV per nucleon. However, a reaction of this kind would require the simultaneous collision of four nuclei, a process which is so highly improbable at normal densities that it practically does not occur. Indeed, as we have seen before, the fusion of two reaction partners is already a rather improbable process, so only two particle collisions can be envisaged for fusion reactions in a reactor.

Altogether more than 80 different fusion reactions are currently known. We are mainly interested in the D-T reaction (2). It is accompanied by a number of side-reactions, the most important among which are D-D and T-T reactions. It is, however, possible to neglect these side reactions, and we shall therefore not consider them.

In the D-T reaction the main portion of the energy is released to the neutron. Although the fast fusion neutrons created this way lead to secondary radioactivity in some materials surrounding the plasma (first wall, supports, etc.), in magnetic confinement schemes this must, at least at present, be considered an advantage. Since they don't carry electric charge the neutrons are not held back by the confining magnetic field, and they can also easily penetrate the confinement vessel. Outside of this their energy can be extracted by a moderator.

II. CROSS-SECTIONS, REACTION RATES AND POWER DENSITY OF FUSION REACTIONS

A fusion reaction which releases a lot of energy but occurs very rarely is of little use. Thus the reaction frequency is a crucial issue. Let us consider a beam of D-nuclei with density n_{D}, moving at constant relative velocity v through T-nuclei. The number dn_{D} of beam particles that is lost due to interaction processes such as scattering collisions or fusion reactions when the beam advances by a distance ds is proportional to ds, to the density n_{T} of target particles and to that of the beam particles, n_{D}:

$$\begin{aligned} dn_{\mathrm{D}} &= \sigma n_{\mathrm{D}} n_{\mathrm{D}}\, ds \quad \Rightarrow \\ R &:= \dot n_{\mathrm{D}} = \dot n_{\mathrm{T}} = n_{\mathrm{D}} n_{\mathrm{T}} \langle \sigma(v) v \rangle , \end{aligned} \tag{4}$$

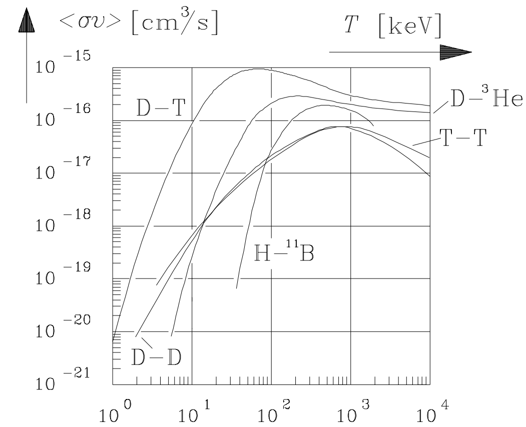

Figure 2: Temperature dependence of $\langle \sigma v \rangle$.

where $\langle\,\rangle$ denotes the averaging over particles of all possible velocities. R is the *reaction rate* (or the collision frequency in the case of scattering collisions; for both processes independently a corresponding equation applies). The evaluation of the average $\langle \sigma(v) v \rangle$ requires some thermodynamics, involves the Gamow factor, and yields the results shown in Fig. 2 for some typical fusion reactions. It is seen that it assumes by far the largest values in the D-T reaction, and this even at much lower temperatures than in the other fusion reactions.

It is only a small fraction of highly energetic particles that are reacting and being lost through fusion. This tail is repopulated by scattering collisions that cause the plasma to approach a Maxwellian distribution as closely as possible. This collisional process for the replacement of highly energetic particles lost by fusion is an essential characteristic of thermonuclear fusion. Thus while scattering collisions have the unpleasant side effect of causing diffusion and particle losses from the reaction vessel on the one hand, on the other hand they have the important task of replenishing highly energetic particles lost by fusion. In a fusion reactor each fusion collision will be accompanied by a sufficiently high number of scattering collisions. Closer investigation shows that at the temperature of a fusion reactor (≈ 10 keV) on average for each fusion collision there are about 8000 scattering collisions.

The quantity which characterizes the efficiency of a fusion reaction is the power density P_{fus}, the energy released per second in a unit volume:

$$P_{\mathrm{fus}} = R E_{\mathrm{fus}} = n_{\mathrm{D}} n_{\mathrm{T}} \langle \sigma v \rangle E_{\mathrm{fus}} , \quad E_{\mathrm{fus}} = 17.6\ \mathrm{MeV} . \tag{5}$$

Both ions and electrons exert a pressure p_{I} and p_{e} adding to a total pressure p,

$$p_{\mathrm{I}} = n_{\mathrm{I}} k T_{\mathrm{I}} = (n_{\mathrm{D}} + n_{\mathrm{T}}) k T_{\mathrm{I}} , \tag{6}$$

$$p_{\mathrm{e}} = n_{\mathrm{e}} k T_{\mathrm{e}} , \quad p = p_{\mathrm{I}} + p_{\mathrm{e}} . \tag{7}$$

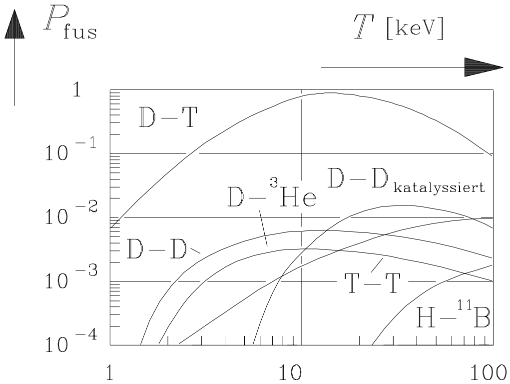

Figure 3: Maximum fusion power as function of T.

Due to stability reasons, there is an upper limit to the ratio

$$\beta = \langle p\rangle/\langle B^2/2\mu_0\rangle \tag{8}$$

that is $\beta_{\max} \approx 6\,\%$ in tokamaks. From this and (6)–(7) it follows that there is an upper limit of the fusion particle density that for $n_D = n_T = n_I/2$ and $n_e = n_D + n_T$ (quasi-neutrality) is given by $\langle \widehat{n}_I\rangle \approx \langle B^2\beta_{\max}/(8\mu_0 kT)\rangle$. The maximum fusion power associated with this is

$$\widehat{P}_{fus} = \left\langle \frac{B^4\beta^2}{64\mu_0{}^2k^2}\frac{\langle\sigma v\rangle}{T^2}E_{fus}\right\rangle . \tag{9}$$

Fig. 3 shows $\widehat{P}_{fus}$ for given $\beta_{\max}$ as a function of the temperature for several fusion reactions. Comparison with Fig. 2 reveals that the greatest power output is obtained at a much lower temperature than the maximum value of $\langle\sigma v\rangle$. This is due to the factor $1/T^2$ in $\widehat{P}_{fus}$ that, with increasing temperature, causes $\widehat{P}_{fus}$ to decrease before $\langle\sigma v\rangle$ has reached its maximum value.

III. BALANCE EQUATIONS

A. Particle balance

A general particle balance equation has the form

$$\partial n_k/\partial t + \mathrm{div}(n_k \boldsymbol{v}_k) = Q_k\,. \tag{10}$$

$n_k\boldsymbol{v}_k$ is the current of particles, and Q_k is a local source term. The equation accounts for (a) particle supply, (b) particle gains and losses through the burning of the fuel, and (c) for losses by diffusion. Averaging eq. (10) over the whole plasma volume V yields

$$d\overline{n}_k/dt + \int n_k\boldsymbol{v}_k\cdot d\boldsymbol{S}/V = \overline{Q}_k\,, \tag{11}$$

where $\overline{n}_k = \int n_k\,dV/V$ is the average particle density and $\overline{Q}_k = \int Q_k\,dV/V$ the average source term.

The latter is composed of the fuel losses described by (4) and a term $\overline{s}_k$ accounting for the fuel supply: $\overline{Q}_i = -\overline{n_i n_j\langle\sigma v\rangle} + \overline{s}_i$. After division by V the total loss of particles per second from the plasma, $dN_k^{\mathrm{loss}}/dt = \int n_k\boldsymbol{v}_k\cdot d\boldsymbol{S}$, yields an average diffusion loss rate per volume. This leads to the definition of a particle loss time τ_k through

$$\tau_k = \frac{\overline{n}_k V}{\int n_k\boldsymbol{v}_k\cdot d\boldsymbol{S}}\,. \tag{12}$$

Its precise meaning can be seen from the reformulation $\tau_k\int_{S_V} n_k\boldsymbol{v}_k\cdot d\boldsymbol{S} = \tau_k\,dN_k^{\mathrm{loss}}/dt = \overline{n}_k V = N_k$: Under stationary conditions obtained when all diffusion losses are compensated for by supply, τ_k is the time elapsed until just as many particles are lost from the plasma through diffusion as it momentarily contains. (We assume that effects of particle recycling[1] are included in τ_k.)

Assuming approximately equal diffusion loss times, $\tau_i = \tau_j = \tau_p$, and using the approximations

$$\overline{n_i n_j\langle\sigma v\rangle} \approx \overline{n_i}\,\overline{n_j}\,\overline{\langle\sigma v\rangle}\,, \quad \overline{\langle\sigma v\rangle(T)} \approx \langle\sigma v\rangle(\overline{T})\,, \tag{13}$$

from (11) we obtain the burn equations

$$dn_i/dt = -n_i/\tau_p - n_i\,n_j\,\langle\sigma v\rangle + s_i\,, \tag{14}$$

where the volume-averaging bars have been omitted for further convenience. These "zero-dimensional" equations can be improved by taking into account profile effects: For profiles of a given (not self-consistently determined) shape each term is modified by a shape factor (see e.g. Ref. 2).

Eqs. (14) must be supplemented by the *quasi-neutrality condition*

$$\sum n_k Z_k = n_e \tag{15}$$

in which Z_k is the charge number of ion species k.

B. Energy balance

With the simplifying assumption $T_e = T_D = T_T = T$ (this implies that all ions created by fusion are thermalized) the general energy balance equation has the form

$$\frac{\partial}{\partial t}\tfrac{3}{2}(n_e + \sum_\lambda n_\lambda)kT + \mathrm{div}\boldsymbol{J} = P_{OH} + P_{ext} + P_\alpha - P_{rad} \tag{16}$$

where $\boldsymbol{J}$ is the total heat flow current due to heat convection and heat conduction, P_{OH} is the Ohmic heating power, P_{ext} additional external heating power, P_α the alpha particle heating power; the work $\boldsymbol{v}_e\cdot\boldsymbol{\nabla}p_e + \sum_\lambda \boldsymbol{v}_\lambda\cdot\boldsymbol{\nabla}p_\lambda$ performed by the pressures has been neglected in comparison with the much larger heat source terms.

1. Ohmic heating: At fusion temperatures $P_{OH} = \eta j^2$ can usually be neglected in comparison with P_α since $\eta \propto 1/T^{3/2}$. (This would not be possible in tokamaks with extremely strong magnetic fields because in these much stronger currents would be allowed.)

2. External heating: It is useful to express the external heating power as a fraction of the fusion power through

$$P_{\rm ext} = P_{\rm fus}/Q = 5P_\alpha/Q\,. \tag{17}$$

Q is called the *power enhancement factor*.

3. Alpha particle heating: In our calculations we shall assume that the energy released to the alpha particles through fusion processes is fully delivered to the plasma. The heating power thus obtained is approximately given by (5) with $E_{\rm fus}$ replaced by E_α, i.e.

$$P_\alpha = n_{\rm D} n_{\rm T} \langle\sigma v\rangle E_\alpha = n_{\rm D} n_{\rm T} \langle\sigma v\rangle E_{\rm fus}/5\,. \tag{18}$$

This is only an approximation for the following reasons:
1. Some alpha particles may already diffuse out, before they have delivered their surplus energy to the plasma.
2. Our expression for P_α is a function of the position and time of alpha particle creation; however, the real position and time of energy deposition are somewhat apart or later respectively.
Due to $P_\alpha = P_{\rm fus}/5$ the temperature dependence of P_α is the same as that of $P_{\rm fus}$ shown in Fig. 3.

4. Radiation losses: There are radiation losses through bremsstrahlung, synchrotron radiation, and through line and recombination radiation. At the temperatures of a D-T reactor, $10-20$ keV, synchrotron radiation can be neglected in comparison with bremsstrahlung. For bremsstrahlung we shall employ the formula

$$P_{\rm B} = \frac{e^6}{24\pi\varepsilon_0{}^3 c^3 m_{\rm e} h}\, n_{\rm e}{}^2 Z^2 \sqrt{\frac{8kT_{\rm e}}{\pi m_{\rm e}}}\, g_{\rm ff}\left(\frac{Z^2}{T_{\rm e}}\right), \tag{19}$$

in which $g_{\rm ff}$ is a slowly varying function of its argument called *Gaunt-factor*. In the D-T reaction there are separate contributions of this kind from D and T with $Z=1$ and from ^{4}He with $Z=2$.

In a pure D-T plasma line and recombination radiation do not play an essential role except for the much cooler plasma boundary region, because all ions are fully ionized and the central plasma is too hot for recombinations. The situation is different if the plasma is polluted by nuclei of higher charge number. We shall only rather crudely take into account such radiation, employing for it again eq. (19) with some effective charge number for the impurities.

5. Transport losses: Integrating the heat flow $\boldsymbol{J}$ through diffusion and convection of energy over the plasma boundary yields the total energy losses by transport. In analogy to (12) we introduce an energy confinement time τ_E through

$$\tau_E = \frac{\int_V \frac{3}{2}(n_{\rm e} + \sum_\lambda n_\lambda)\,dV}{\int_{S_V} \boldsymbol{J}\cdot d\boldsymbol{S}}\,. \tag{20}$$

Frequently, especially by experimentalists, a different energy confinement time τ_E^* is used that is defined through

$$\tau_E^* = \frac{\int_V \frac{3}{2}(n_{\rm e} + \sum_\lambda n_\lambda)\,dV}{P_{\rm rad} + \int_{S_V} \boldsymbol{J}\cdot d\boldsymbol{S}}\,. \tag{21}$$

It is the time in which the plasma, due to all losses including radiation, loses the same amount of energy as it presently contains and is easier to measure than τ_E.

6. Averaged energy balance equation: Integrating eq. (16) over the whole plasma volume, dividing by V and using (17), (20) plus the same approximations as in (13), with omission of the bar for averages we obtain

$$\frac{d}{dt} e_{\rm tot} = P_\alpha\left(1+\frac{5}{Q}\right) - \frac{e_{\rm tot}}{\tau_E} - P_{\rm rad} = P_\alpha\left(1+\frac{5}{Q}\right) - \frac{e_{\rm tot}}{\tau_E^*} \tag{22}$$

where $e_{\rm tot} = (n_{\rm e} + \sum_\lambda n_\lambda)kT$ is the total energy density and the expressions for P_α and $P_{\rm rad}$ must be evaluated at the average temperature and density.

C. Basic equations for the D-T reaction

We now make a further approximation in neglecting all side reactions (D-D, T-T etc.) due to their small fusion cross-sections. With this we obtain the particle balance equations

$$\frac{dn_{\rm D}}{dt} = -\frac{n_{\rm D}}{\tau_{\rm p}} - n_{\rm D} n_{\rm T}\langle\sigma v\rangle_{\rm DT} + s_{\rm D}\,, \tag{23}$$

$$\frac{dn_{\rm T}}{dt} = -\frac{n_{\rm T}}{\tau_{\rm p}} - n_{\rm D} n_{\rm T}\langle\sigma v\rangle_{\rm DT} + s_{\rm T}\,, \tag{24}$$

$$\frac{dn_\alpha}{dt} = -\frac{n_\alpha}{\tau_\alpha} + n_{\rm D} n_{\rm T}\langle\sigma v\rangle_{\rm DT}\,, \tag{25}$$

and the energy balance equation

$$\begin{aligned} &\frac{d}{dt}\left[\tfrac{3}{2}(n_{\rm e} + n_{\rm I} + n_\alpha + n_Z)kT\right] \\ &\quad = -\tfrac{3}{2}(n_{\rm e} + n_{\rm I} + n_\alpha + n_Z)kT/\tau_E \\ &\qquad + n_{\rm D} n_{\rm T}\langle\sigma v\rangle_{\rm DT} E_\alpha (1+5/Q) - P_{\rm B} \end{aligned} \tag{26}$$

where n_Z is the density of impurity ions considered as a single species with effective charge number Z. We shall consider n_Z as a given parameter. In contrast to our previous intentions, we have introduced a separate particle confinement time $\tau_\alpha \neq \tau_{\rm p}$ for the alpha particles, the purpose being that this will facilitate the transition to a limiting case to be considered. In addition, we have the quasi-neutrality condition

$$n_{\rm I} + 2n_\alpha + Zn_Z = n_{\rm e} = n_{\rm tot}/2 \,. \qquad (27)$$

For $P_{\rm B}$ we have to take into account the radiation caused by hydrogen isotopes ($Z = 1$), alpha particles ($Z = 2$) and impurities (charge number Z), from (19) obtaining the formula

$$P_{\rm B} = n_{\rm e}{}^2 \left[c_{\rm I} R_{\rm I}(T) + c_\alpha R_\alpha(T) + c_Z R_Z(T) \right] \qquad (28)$$

in which we employed the concentrations

$$c_{\rm I} = n_{\rm I}/n_{\rm e} \,, \quad c_\alpha = n_\alpha/n_{\rm e} \,, \quad c_Z = n_Z/n_{\rm e} \qquad (29)$$

and where

$$\begin{aligned} R_{\rm I} &= C_{\rm B}\sqrt{T}\, g_{\rm ff}(1/T) \,, \quad R_\alpha = 4C_{\rm B}\sqrt{T}\, g_{\rm ff}(4/T) \,, \\ R_Z &= Z^2 C_{\rm B}\sqrt{T}\, g_{\rm ff}\left(Z^2/T\right) \end{aligned} \qquad (30)$$

with

$$C_{\rm B} = \frac{e^6\sqrt{8k}}{24\pi\varepsilon_0{}^3 c^3 m_{\rm e} h \sqrt{\pi m_{\rm e}}} \,. \qquad (31)$$

IV. EQUILIBRIA: BREAK-EVEN AND IGNITION

We now want to determine equilibria, i.e. we are looking for stationary solutions ($d/dt = 0$). When an equilibrium is achieved with $P_{\rm ext} = P_{\rm fus}$ or $Q = 1$ resp. this is called *break-even*. *Ignition* is achieved when all external heat sources can be turned off, $P_{\rm ext} = 0$ or $Q = \infty$. From (23)–(24) for stationary conditions we get

$$s_{\rm D} = \frac{n_{\rm D}}{\tau_{\rm p}} + n_{\rm D} n_{\rm T} \langle\sigma v\rangle_{\rm DT} \,, \quad s_{\rm T} = \frac{n_{\rm T}}{\tau_{\rm p}} + n_{\rm D} n_{\rm T} \langle\sigma v\rangle_{\rm DT} \,, \qquad (32)$$

the magnitude of the particle sources is fixed by the requirement of stationarity. The maximum fusion power is obtained for $n_{\rm D} = n_{\rm T} = n_{\rm I}/2$, i.e. the particle sources must satisfy

$$s_{\rm D} = s_{\rm T} = \frac{n_{\rm I}}{2\tau_{\rm p}} + \frac{n_{\rm I}{}^2}{4} \langle\sigma v\rangle \,. \qquad (33)$$

With this equations (23) and (24) are satisfied and must no longer be considered concerning equilibrium.

The remaining equations to be solved are (25), (26) and (15) viz.

$$c_{\rm I} + 2c_\alpha + Zc_Z = 1 \,. \qquad (34)$$

The latter one is satisfied when we eliminate $c_{\rm I}$ by using $c_{\rm I} = 1 - 2c_\alpha - Zc_Z$. Inserting this we are left with only two equations,

$$\frac{c_\alpha n_{\rm e}}{\tau_\alpha} = \tfrac{1}{4}(1 - 2c_\alpha - Zc_Z)^2 n_{\rm e}{}^2 \langle\sigma v\rangle \,, \qquad (35)$$

$$\begin{aligned} &\tfrac{3}{2}[2 - c_\alpha - (Z-1)c_Z] n_{\rm e} kT/\tau_E \,, \qquad (36) \\ &\quad = \tfrac{1}{4}(1 - 2c_\alpha - Zc_Z)^2 n_{\rm e}{}^2 \langle\sigma v\rangle E_\alpha (1 + 5/Q) - P_{\rm B} \,. \end{aligned}$$

A. Ideal ignition condition, minimum burn temperature, and ideal break-even

In a first quantitative approach we shall neglect the presence of impurities as well as that of the helium ash, i.e. we set $c_\alpha = 0$, $c_Z = 0$. This way we not only get a widely used result for the ignition condition but also one which is very easily comprehensible. Of course this can only be a rough approximation because the accumulation of helium ash can, in principle, not be avoided. $c_\alpha = 0$ is compatible with the equilibrium equations if in (35) we set $\tau_\alpha = 0$ and don't consider this equation any longer. (This is the reason why we introduced a separate confinement time τ_α.) For the bremsstrahlung we have $P_{\rm B} = n_{\rm e}{}^2 R_{\rm I}(T)$, and the only equation left is the energy equation (36) which becomes

$$3n_{\rm e}kT/\tau_E = \tfrac{1}{4} n_{\rm e}{}^2 \langle\sigma v\rangle E_\alpha (1 + 5/Q) - n_{\rm e}{}^2 R_{\rm I}(T) \,. \qquad (37)$$

Dividing it by $n_{\rm e}{}^2$ and then solving it with respect to $n_{\rm e}\tau_E$ we finally obtain the *ideal ignition criterion*

$$n_{\rm e}\tau_E = \frac{3kT}{\frac{1}{4}\langle\sigma v\rangle E_\alpha (1 + 5/Q) - R_{\rm I}(T)} \,. \qquad (38)$$

The product $n_{\rm e}\tau_E$ is a measure of the quality of the plasma confinement, and the value required according to this formula in order to get an ignited equilibrium or break-even depends only on the temperature. This temperature dependence is shown in Fig. 4. The minimum temperature required for ignition ($Q = \infty$) is obtained by equating the denominator of our result for $n_{\rm e}\tau_E$ to zero (τ_E becoming infinite). It is given by the smaller temperature obtained as a solution from

$$R_{\rm I}(T) = \tfrac{1}{4}\langle\sigma v\rangle E_\alpha \qquad (39)$$

and is typically about 6 keV.

We shall now transform our ideal ignition curve into a diagram employing our second energy confinement time τ_E^* defined in (21). Applying this definition,

$$P_{\rm rad} + e_{\rm tot}/\tau_E = e_{\rm tot}/\tau_E^* \,, \qquad (40)$$

to our present situation yields

$$n_{\rm e}{}^2 R_{\rm I} + 3n_{\rm e}kT/\tau_E = 3n_{\rm e}kT/\tau_E^* \qquad (41)$$

and

$$\tau_E = \frac{3kT\tau_E^*}{3kT - n_{\rm e}\tau_E^* R_{\rm I}} \,. \qquad (42)$$

Since τ_E must be nonnegative, from this we get the condition

$$n_{\rm e}\tau_E^* \le 3kT/R_{\rm I} \,. \qquad (43)$$

The limit $n_{\rm e}\tau_E^* = 3kT/R_{\rm I}$ is called *radiation limit* because $\tau_E = \infty$ for it, and all losses are due to radiation. With (42)

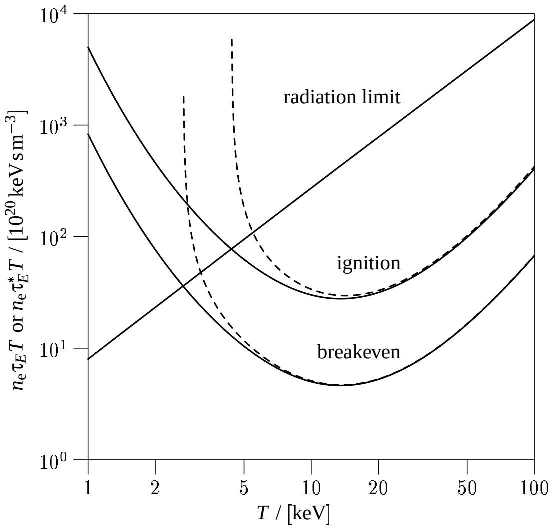

Figure 4: Curves $n_e\tau_E T = f_1(Q;T)$ (dashed lines) and $n_e\tau_E^* T = f_2(Q;T)$ (solid lines) for ideal ignition and ideal break-even. Also shown is the radiation limit (only relevant for the description by τ_E^*). (Figure adapted from Ref. 5.)

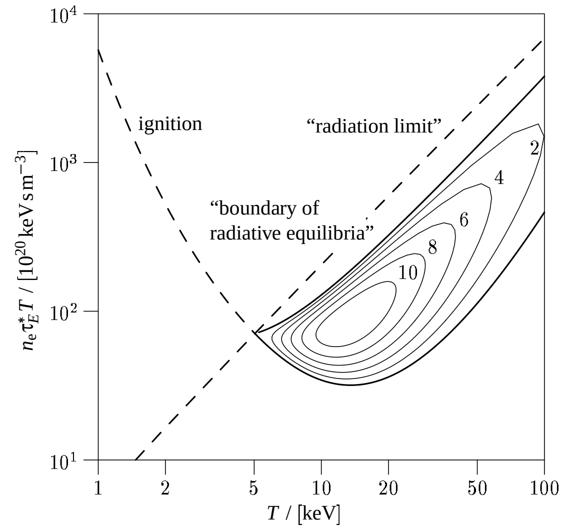

Figure 5: Curves $n_e\tau_E^* T = f(T)$ for non-ideal ignited equilibria, radiation limit, and boundary of radiative equilibria assuming an impurity concentration of $f_Z = 2\%$ beryllium ($Z = 4$). (Figure adapted from Ref. 5.)

and multiplication by T our condition (38) transforms into

$$n_e\tau_E^* T = \frac{12kT^2}{\langle\sigma v\rangle E_\alpha(1+5/Q)}\,. \tag{44}$$

The so-called *fusion product* $n_e\tau_E^* T$ employed in this formula is widely used for characterizing the performance of a fusion device because it combines the two quantities $n_e\tau_E^*$ (also a measure for the quality of confinement) and T, which both have to be large for ignition, into a single quantity. Its temperature dependence is shown in Fig. 4 together with the radiation limit. According to (43) only states below the radiation limit are physically meaningful. The two intersection points between the radiation limit and the ignition curve describe *radiative equilibria*. The temperature at the left point is the minimum temperature for which ignition is possible.

In the temperature range of a fusion reactor a good approximation for $\langle\sigma v\rangle$ is provided by[3]

$$\langle\sigma v\rangle = 1.1\cdot 10^{-24} T^2 \text{ m s}^{-1}\,, \quad T \text{ in keV}\,. \tag{45}$$

Inserting this in (44) yields the ideal conditions $n_e\tau_E^* T = 3\cdot 10^{21}\,\text{m}^{-3}\,\text{keV s}$ for ignition, e.g. reached with $n = 10^{20}\ \text{m}^{-3}$, $T = 10$ keV and $\tau_E^* = 3$ s, and $n_e\tau_E^* T = 0.5\cdot 10^{21}\,\text{m}^{-3}\,\text{keV s}$ for break-even which has already been reached in two tokamak devices (JET and JT-60).

B. Non-ideal ignition and break-even

We shall now discuss the influence of the helium ash and impurities on the conditions for ignition and break-even. For $\tau_\alpha \neq 0$ from (35) we also get $c_\alpha \neq 0$, and since according to (15) each α-particle displaces two fuel particles and according to (19) radiates twice as much as the two together, too high alpha particle concentrations will inevitably cause the nuclear fire to suffocate. Thus, welcome as they are with respect to heating, the alpha particles may lead to a dangerous fuel dilution and provide a rather unpleasant pollution if they become too numerous. It is therefore important that they disappear due to diffusion and convection, thereby unfortunately being accompanied by fuel particles.

Diffusion and convection are the only loss mechanisms for particles, and there is no mechanism that could be compared with the loss of energy by radiation. Although the mechanisms of particle and of energy diffusion are quite different, there is a strong coupling between them. The scaling ansatz[4,5]

$$\tau_p/\tau_E = \rho = \text{const}\,. \tag{46}$$

appears as a good approximation for the helium ash particles in the plasma core because in this particular case (distinct from other species in the plasma) the particle and energy source profiles are identical. Since particles are somewhat better confined than energy, a value $\gtrsim 5$ is expected for the ratio.

The statements made above can now be quantified by solving eqs. (35)–(36) together with (28) and the scaling ansatz (46). After c_α is eliminated from the equations, one can again derive an equation for $n_e\tau_E T$, this time as a function of T and ρ, that can be put into the form[5]

$$\rho = \rho(n_e\tau_E T, T)\,. \tag{47}$$

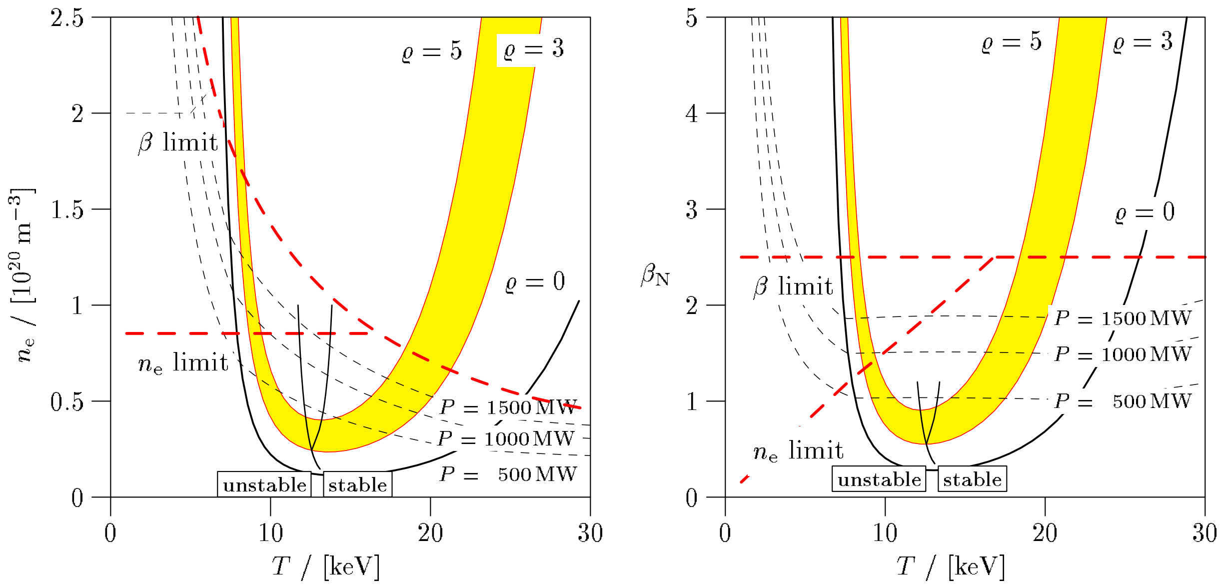

Figure 6: Ignition curve $\rho = 5$ (together with curves $\rho = 3$ and $\rho = 0$ for comparison) (a) in a n, T-plane and (b) in a β_N, T-plane. ($\beta_N = \beta/(I/aB)$ is the so-called normalized beta.) In both diagrams the boundary of stability with respect to thermal instabilities is also shown. The stable regime is to the right of all stability curves. (Figures taken from Ref. 2.)

Fig. 5 shows the ignition curves $\rho = \text{const}$ numerically obtained from this for $Q = \infty$. For $\rho = 0$ (corresponding to $\tau_\alpha = 0$) our previous ideal curves are recovered. For $\rho > 0$ one obtains closed ignition curves, and it was shown in Ref. 5 that one also obtains closed ignition curves for $n_e\tau_E^* T(T)$ if the scaling assumption (46) with τ_E is being kept. The most important outcome of these calculations is that ignited equilibria exist only for $\rho \lesssim 15$ (or $\rho \lesssim 10$ for an impurity concentration of 2% beryllium). If ρ becomes larger, the helium concentration becomes too large and ignition impossible as predicted before by our qualitative arguments. The helium concentration at which this occurs is about 30 percent.

V. ITER CONFINEMENT SCALING LAWS AND TRANSFORMATION OF IGNITION CURVES TO THE n, T AND β, T PLANE

For the planning of an ignition experiment like ITER it is important to have some idea about what confinement properties may be expected. Theoretically plasma transport is a very difficult and not yet satisfactorily solved problem, so the answers to this question must be essentially extrapolated from experimental data. In huge international databases the transport properties of many different tokamaks under many different circumstances have been collected and evaluated, applying as constraints certain theoretical criteria.[3,6,7] One expects that the energy confinement time τ_E will depend on design parameters according to scaling laws such as ITER 89-P,[8]

$$\tau_E = 0.048\, f_H M^{0.5} I^{0.85} B^{0.2} R^{1.2} a^{0.3} \kappa^{0.5} P^{-0.5} n_e^{0.1}\,, \qquad (48)$$

where f_H is the H-mode enhancement factor ($f_H = 2.0$ in Fig. 6), M the isotopic mass ($= 2.5$ for a 50:50 D-T mixture), I the plasma current in MA, B the toroidal magnetic field in Teslas, R and a the major and minor tokamak radius in meters, κ the elongation of the plasma cross-section, n_e the electron density in $10^{20}\,\text{m}^{-3}$, and $P = P_{OH} + P_{ext} + P_\alpha$ the net heating power in MW. Using the equilibrium equation (36), P can be replaced by $\frac{3}{2} n_{tot} T/\tau_E$ and (48) rewritten as

$$\tau_E = \left(0.048\, f_H M^{0.5} I^{0.85} B^{0.2} R^{1.2} a^{0.3} \kappa^{0.5}\right)^2 n_e^{-0.8} T^{-1.0}\,. \qquad (49)$$

With this relation the ignition contours (47) can be translated from the $n_e\tau_E T, T$-plane directly into the n_e, T-plane (for details see Ref. 9; note, however, that there the ITER scaling laws where applied to the energy confinement time τ_E^* including radiation losses while according to Ref. 10 they should, in fact, be applied to the confinement time excluding radiation losses). Fig. 6(a) shows the "ignition curve" $\rho = 5$ (together with $\rho = 3$ and $\rho = 0$ for comparison) in a n_e, T-plane, and using $\beta = n_{tot} kT B^2/(2\mu_0)$ a similar diagram can be obtained in the β, T-plane (see Fig. 6(b)). The advantage of representing the ignition curve as $n_e = n_e(T)$ or $\beta = \beta(T)$ is that the impact of plasma stability limits like the β-limit or the Greenwald density limit can immediately be seen.[11]

VI. BURN STABILITY

In order to determine the stability of our burn equilibria with respect to thermal instabilities one has to solve the time dependent equations (23)–(27) for perturbations of the equilibrium states. This has extensively been done in Ref. 9, and here only the most important results are quoted. One problem arising in this context is, how to treat the confinement times during the evolution of instabilities. One possibility would be to keep them constant at their equilibrium values. Since the typical growth times of instabilities turn out to be several seconds under this assumption, it appears reasonable to assume the validity of the scaling law (49) also during this time dependent process because the plasma has time enough to adapt to these conditions which were originally derived for equilibrium states. Redoing the stability calculations with these adapted confinement times appreciably changes the stability behavior, which for this case is shown in Figs. 6 (a) and (b). Stable behavior is obtained to the right of the stability boundaries shown in the diagram. States to the left are unstable and undergo a transition to some state on the right branch of the corresponding ignition curve $\rho = \text{const}$.

VII. LAWSON CRITERION AND REACTOR EFFICIENCY CRITERION

If the plasma of a fusion reactor is ignited, this does not imply that there is also a net energy gain, because there are energy losses during the initial heating phase, and also energy is needed for feeding auxiliary devices to keep the reactor running. The first one to consider problems of this kind was Lawson who, in 1957, formulated the so-called *Lawson criterion*.[12] He asked the question: When does a fusion reactor deliver so much energy that it can run self-sustained, i.e. when does it neither need nor deliver energy?

In order to answer this question, we consider the sum of the internal plasma energy and the energy released in the form of radiation and fusion energy during the burn time τ_b, all expressed as specific quantities per volume,

$$e_{th} + e_{rad} + e_{fus}\,, \tag{50}$$

where

$$e_{th} = 3n_e kT\,, \quad e_{rad} = C_B\, g_{ff} n_e^2 \sqrt{T}\,\tau_b\,, \qquad e_{fus} = \tfrac{1}{4} n_e^2 \langle\sigma v\rangle E_{fus}\,\tau_b\,. \tag{51}$$

This sum of energies is converted with efficiency η_{th}, and in a self-sustained power station it supplies the thermal energy of the plasma and the radiation losses:

$$(e_{th} + e_{rad} + e_{fus})\,\eta_{th} = e_{th} + e_{rad}\,. \tag{52}$$

After the explicit expressions for the different energy terms are inserted, one can solve with respect to $n_e\tau_b$ to obtain the *Lawson criterion*

$$n_e\tau_b = \frac{12kT}{\langle\sigma v\rangle E_{fus}\,\eta_{th}/(1-\eta_{th}) - 4C_B\, g_{ff}\sqrt{T}}\,. \tag{53}$$

Similarly one can ask the question: When does a reactor yield the efficiency η? In order to answer this question we assume a pulsed operation of the reactor with a start-up phase of duration τ_h for heating the plasma to ignition, and a burning time τ_b with stationary conditions at temperature T. In the start-up phase for each volume element a heating energy e_h must be supplied externally, from which a fraction

$$e_a = \eta_a e_h \tag{54}$$

is absorbed by the plasma for providing its internal energy and compensating all heat losses (transport and radiation). The net efficiency of the power station is defined through

$$\eta = e_{net}/e_{fus}\,, \tag{55}$$

where e_{fus} is the total fusion energy gain per volume (at present, probably not all fusion energy delivered to the alpha particles can be envisaged for conversion), and e_{net} is the energy per volume that can be supplied to the mains as electricity. Considering all important energy flows in the reactor station, the following *reactor efficiency criterion* can be derived:

$$n_e\tau_b = \frac{1}{\eta_a\,(\eta_{eff}-\eta)}\,\frac{12kT(1+\tau_h/\tau^*_{E,h})}{\langle\sigma v\rangle\, E_{fus}}\,, \tag{56}$$

where $\eta_{eff} \approx \eta_{th} \approx 1/3$. We can combine this efficiency criterion with the corresponding ideal ignition criterion (44). Dividing the first by the second yields (for $Q = \infty$)

$$\frac{\tau_b}{\tau^*_E} = \frac{(1+\tau_h/\tau^*_{E,h})}{5\eta_a\eta_{eff}\,(1-\eta/\eta_{eff})}\,, \tag{57}$$

where $E_\alpha/E_{fus} = 1/5$ was used. This shows that the factor by which the burning time τ_b must exceed the energy confinement time τ^*_E is independent of the temperature.

Assuming $\tau_h \approx \tau^*_{E,h}$ and $\eta_a\eta_{eff} \approx 1/20$ we get $\tau_b \approx 8\,\tau^*_E/(1-\eta/\eta_{eff})$. For $\eta/\eta_{eff} = 0.95$ this yields $\tau_b \approx 160\,\tau^*_E$ or $\tau_b \approx 560$ s for $\tau^*_E = 3.5$ s as expected in a fusion reactor.

In fact much longer burn times will be required for other reasons: A reactor must last for about 25 years at least in order to repay for the large expenses that are needed for its construction. A burn time of 200 s only would imply about $4 \cdot 10^6$ start-ups and thus changes between hot and cold during its life time. This is more than the reactor will stand according to all technical experience. A reasonable number of changes will be no more than about 100 000. In that case a burning cycle would have to last for about 2 h in order to sum up to a life time of 25 years.

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