REDUCED DESCRIPTIONS FOR COMPLEX PLASMAS: SIMPLE EXAMPLES, PLASMA KINETICS, AND TRANSPORT TRUNCATIONS

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ABSTRACT

The statistical description of a hot, magnetized, and classical plasma is reviewed. The latter represents the appropriate model for a fusion plasma in magnetic confinement. Various approaches for (reduced) kinetic descriptions are presented. We first discuss the problems related with reduction of information by investigating extremely simple mathematical models and reviewing standard projection techniques. The famous Boltzmann equation for dilute gases is then presented (without a systematic derivation), and the differences between the kinetic and the hydrodynamic regimes are worked out. In the main part, the consequences of long-range Coulomb interactions are demonstrated. Several plasma-kinetic equations, like for instance the Balescu-Lenard equation, are are systematically presented. Physical consequences from the linearization of the kinetic equations, e.g. collision frequencies and Landau damping, are elucidated. In the final part of the paper the specific re-formulations in magnetized plasmas are investigated. The drift-kinetic and the gyrokinetic approaches are presented. The paper is concluded by an outlook on often used truncations.

I. INTRODUCTION

A plasma is a many body system consisting of a huge number of interacting particles (in the simplest case electrons and protons; in general we have several different species). In the present paper we investigate a hot, magnetized, classical plasma as the standard model – in the sense of theoretical physics – for a fusion plasma appearing in magnetic confinement. The approximations characterized by hot, magnetized, and classical have been discussed in previous Carolus Magnus lectures. They are not satisfied for all fusion plasmas, especially when inertial confinement is considered. However, for a tokamak or stellarator plasma they are good approximations in a broad range of applications.

The particles of a classical plasma follow the classical laws of non-relativistic dynamics, and the exact formulation of the problem is straightforward. When the position vector of the i-th particle of species s is denoted by $\vec{r}_i^{\,s}$, we immediately can write the equations of motion,

$$m^s \, \frac{d^2\vec{r}_i^{\,s}}{dt^2} = q^s \vec{E}(\vec{r}_i^{\,s}, t) + \frac{q^s}{c} \, \frac{d\vec{r}_i^{\,s}}{dt} \times \vec{B}(\vec{r}_i^{\,s}, t) \;, \qquad (1)$$

where the electric and magnetic fields $\vec{E}$ and $\vec{B}$, respectively, follow from Maxwell's equations (in Gaussian units)

$$\nabla \cdot \vec{E} = 4\pi\rho \quad , \quad \nabla \cdot \vec{B} = 0 \;, \qquad (2)$$

$$\nabla \times \vec{E} = -\frac{1}{c}\,\frac{\partial \vec{B}}{\partial t} \quad , \quad \nabla \times \vec{B} = \frac{4\pi}{c}\,\vec{j} + \frac{1}{c}\,\frac{\partial \vec{E}}{\partial t} \;. \qquad (3)$$

Here,

$$\rho \equiv \rho(\vec{r}, t) = \sum_s q^s \sum_{i=1}^{N_s} \delta\left(\vec{r} - \vec{r}_i^{\,s}(t)\right) \;, \qquad (4)$$

$$\vec{j} = \vec{j}(\vec{r}, t) = \sum_s q^s \sum_{i=1}^{N_s} \frac{d\vec{r}_i^{\,s}}{dt}\, \delta\left(\vec{r} - \vec{r}_i^{\,s}(t)\right) \;. \qquad (5)$$

Of course, it may be advisable to subdivide currents and fields into internal and external ones, but this is not the principal point of the present consideration. For an estimate, let us consider a fusion plasma of 5 m^3 volume. It contains approximately 10^{21} particles. As we have shown above, the interaction laws, and thereby the equations of motion, are known. Nevertheless, the solution of the (electrodynamical and) mechanical problem of the dynamics of, e.g., 10^{21} classical "point particles" with fixed masses and charges is hopeless. But even if, from the point of computer times, a solution would be possible, the problem would not be tractable since we do not know all the initial positions of the 10^{21} particles. But even if we ignore that argument, we would get a flood of information which we could not handle in detail. To build averages out of all the data would be the proper choice. But then we may ask the question: Why do we not aim directly for dynamical equations for the averages themselves? This is the key question leading to the development of physical kinetics. Statistical physics is the adequate discipline to develop closed equations for the reduced information.

A hot and magnetized plasma intended for nuclear fusion is being operated (mostly in a non-stationary regime) with external sources of (free) energy. Thus it is not in equilibrium and therefore we cannot use (equilibrium) thermodynamics for a proper description. A non-equilibrium statistical description is needed.

The general idea of all kinetic or (magneto-)hydrodynamic descriptions is to reduce the number of variables, without loosing that information on which one is specifically interested in. Although, in the formulation presented above, we start from a reversible (Hamiltonian) system, in the (reduced) kinetic description we shall find irreversibility, damping etc. Where these new effects enter the scenario is very often hard to detect during the reduction procedure. Therefore it is advisable to sharpen the mind by first investigating some simpler mathematical examples.

II. MATHEMATICAL PRELIMINARIES

When considering a high-dimensional mathematical system, one idea may be to find an optimal system of basis functions such that a truncation resulting in a reduced system based on only a few basis functions gives reasonable results. This is the well-known problem in numerics, but it also appears in transport theory. The question of the role of the unresolved degrees of freedom then remains. Physically, one may also imagine a situation where we have a large-scale motion in the presence of small scale fluctuations, and the chosen basis functions resolve only the large-scale motion. A macroscopic state influenced by turbulent fluctuations may be an application. From experiments we know that very important anomalous dissipative effects may occur. So one issue of the present section is to review a theory which systematically takes care of contributions from the unresolved variables.

In physical kinetics we derive a (closed) kinetic equation for the one-particle distribution function. The distribution function allows us to determine the probability of finding at time t a particle at position $\vec{r}$ with velocity $\vec{v}$. We do not care anymore about positions and velocities of all the other particles. Their effect has been incorporated already (in an averaged form) into the kinetic equation.

II.1 A simple example

Let us exemplify a typical reduction procedure on a simple example first presented by Chorin et al. We start from a (physically) two-dimensional Hamiltonian system

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = p_1 \ , \tag{6}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -x_1(1+x_2^2) \ , \tag{7}$$

$$\dot{x}_2 = \frac{\partial H}{\partial p_2} = p_2 \ , \tag{8}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -x_2(1+x_1^2) \ . \tag{9}$$

These are the canonical equations originating from the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + x_1^2 + x_2^2 + x_1^2 x_2^2) \ . \tag{10}$$

When we intent to resolve only the coordinate x_1 together with the momentum p_1, a naive procedure (similar to the Galerkin truncation) would ignore the coordinate x_2 together with the momentum p_2. The result of a strict (but naive) truncation will be

$$\dot{x}_1 = p_1 \ , \tag{11}$$

$$\dot{p}_1 = -x_1 \ , \tag{12}$$

which are the equations for an undamped harmonic oscillation. No effect of the "second particle" appears, although it is present in the physical system (and interacts with "particle 1").

Now let us assume that "particle 1" is statistically distributed. The probability (for the moment we assume at all times t, later we shall assume a Gibbs distribution only at $t = 0$) is assumed to be given by the equilibrium Gibbs distribution ($\beta = 1/kT \equiv 1$)

$$P = \frac{e^{-H}}{\int e^{-H} dx_1 dp_1 dx_2 dp_2} \ . \tag{13}$$

Then we "average" the first two equations (6) and (7) by operating on them with

$$\hat{P} = \int dx_2 dp_2 \frac{e^{-H}}{\int e^{-H} dx_1 dp_1 dx_2 dp_2} \ . \tag{14}$$

Since

$$\hat{P} x_2^2 = \frac{\int x_2^2 e^{-H} dx_2 dp_2}{\int e^{-H} dx_2 dp_2} = \frac{1}{1+x_1^2} \tag{15}$$

we end up with

$$\dot{x}_1 = p_1 \ , \tag{16}$$

$$\dot{p}_1 = -x_1 \left[1 + \frac{1}{1+x_1^2} \right] \ . \tag{17}$$

These are closed equations for the relevant variables x_1, p_1 in the presence of the unresolved variables x_2, p_2.

II.2 General projection formalism

Now, we formulate the theory more systematically. Suppose we write a system of ODEs in the form

$$\frac{d\varphi}{dt} = R(\varphi) \ , \tag{18}$$

$$\varphi(t=0) = x \ , \tag{19}$$

where φ, R, and x are n-dimensional vectors. The solution of the ODEs will be written as $\varphi(t;x)$ where the argument x indicates the dependence on the initial value(s) x. Formally, we introduce the operator

$$L(x) = R(x) \frac{\partial}{\partial x} \tag{20}$$

and write in semi-group notation the solution as

$$\varphi(t;x) = e^{tL} x \ . \tag{21}$$

The semi-group notation is being suggested by the relation between the general solution of a *linear* partial differential equation (PDE)

$$\frac{\partial u(x,t)}{\partial t} = R(x) \ \frac{\partial u(x,t)}{\partial x} \tag{22}$$

to the ODE (18). It can be proven that the solution of (22) with

$$u(x, t=0) = g(x) \tag{23}$$

is

$$u(x,t) = g(\varphi(t)) \ , \tag{24}$$

where φ satisfies exactly (18) and (19). Writing the PDE (22) in the form

$$\frac{\partial u(x,t)}{\partial t} = Lu \tag{25}$$

the notation

$$u(x,t) = g(\varphi(t;x)) \equiv \left(e^{tL} g \right)(x) \tag{26}$$

becomes obvious. The operator e^{tL} advances x to $\varphi(t;x)$. Comparing the left-hand-side of (22), i.e.

$$\dot{u} = g_\varphi \dot{\varphi} \ , \tag{27}$$

where the dot designates the time-derivative, and we have introduced $g_\varphi = \partial g(\varphi)/\partial\varphi$, with the right-hand-side

$$\begin{aligned} \dot{u} e^{tL} L g &= e^{tL} R(x) \ \frac{\partial}{\partial x} \ g(x) = R(x) \ \ \frac{\partial}{\partial x} \ g(x) \Big|_{x \to \varphi(t;x)} \\ &= R(\varphi) g_\varphi \ , \end{aligned} \tag{28}$$

we recognize the determination of $\varphi(t;x)$ via

$$\frac{d\varphi}{dt} = R(\varphi) \ . \tag{29}$$

Let us now come back to Eqs. (18) and (19) and their formal solution (21).

We designate the first $m(< n)$ components of the initial value vector x by $\hat{x}$ and the components from $m+1$ to n by $\tilde{x}$, i.e.

$$x_j = \begin{cases} \hat{x}_j & \text{for} \quad j = 1, \ldots, m \,, \\ \tilde{x}_j & \text{for} \quad j = m+1, \ldots, n \,. \end{cases} \tag{30}$$

The values with a dash ($\wedge$) are called resolved values whereas the others (with a wiggle $\sim$) are denoted as unresolved values.

The idea is that the full information of the system of ODEs and their initial values is not needed (or not available). Let the dashed *initial* values be precisely known: for the other $n-m$ ones we assume that only a statistical information is available. Therefore we proceed by resolving the dynamics only in the first m *variables* $\varphi_j \equiv \hat{\varphi}_j, j = 1, \ldots, m$. But note that the statistics is caused by *unresolved initial values* $x_j \equiv \tilde{x}_j, j = m+1, \ldots, n$, and we extent the notation to the unresolved variables $\varphi_j \equiv \tilde{\varphi}_j, j = m+1, \ldots, n$. The final result should be a closed system of m ODEs for the resolved variables $\hat{\varphi}_j$ in dependence of the resolved initial values $\hat{x}_j$, i.e. we look for solutions $\hat{\varphi} \equiv \hat{\varphi}(t; \hat{x})$.

The derivation uses projection operators: Functions in L^2 are projected onto the space $\hat{L}^2$ of functions of the m-dimensional vector $\hat{x}$. Several different projections P are used. Let us mention two of them which have counterparts in physical kinetics and transport theory, respectively.

First, consider the orthogonal projection of f onto the span of all functions of $\hat{x}$, given by

$$(Pf)(\hat{x}) = \frac{\int f(x)\rho(x)d\tilde{x}}{\int \rho(x)d\tilde{x}} \,, \quad d\tilde{x} = dx_{m+1} \ldots dx_n \,. \tag{31}$$

In the language of probability, $(Pf)(\hat{x})$ is the conditional expectation of f given $\hat{x}$ and is denoted by $E[f|\hat{x}]$. $E[f|\hat{x}]$ is the best approximation of f by a function of $\hat{x}$, i.e.

$$E[|f - E[f|\hat{x}]|^2] \leq E[|f - h(\hat{x})|^2] \tag{32}$$

for all functions h in $\hat{L}^2$.

Secondly, pick a set of functions of $\hat{x}$, say $h^\nu(\hat{x}), \nu = 1, \ldots, M$, and make them orthogonal: $(h^\nu, h^\mu) = \delta_{\mu\nu}$. Define a projection

$$(Pf)(\hat{x}) = \sum_{\nu=1}^{M} (f, h^\nu) h^\nu(\hat{x}) \,. \tag{33}$$

Here, $(\ldots, \ldots)$ is the inner product which involves an integration over all x. If the h^ν span L^2 as M increases, the result approximates the conditional expectation $E[f|\hat{x}]$. This is the finite rank projection.

Now let us derive the (closed) system of reduced equations. We write the time-derivative as a partial derivative in order to be able to show also the dependence on the initial value(s). Starting form (21) we find for $j = 1, \ldots, m$

$$\frac{\partial}{\partial t} \varphi_j(t; x) = \frac{\partial}{\partial t} e^{tL} x_j = e^{tL} L x_j \equiv e^{tL} I L x_j \,. \tag{34}$$

The unit operator I is split into

$$I = P + Q \,, \tag{35}$$

where P is the just discussed projection operator that projects functions in L^2 onto the space $\hat{L}^2$ of functions of the m-dimensional vector space $\hat{x}$. Next we introduce the abbreviations

$$\mathcal{R}_j(\hat{x}) = PLx_j \tag{36}$$

and

$$\mathcal{R}_j(\hat{\varphi}(t; x)) = e^{tL} \mathcal{R}_j(\hat{x}) \tag{37}$$

to write

$$\frac{\partial}{\partial t} \varphi_j(t; x) = \mathcal{R}_j(\hat{\varphi}(t; x)) + e^{tL} QLx_j \tag{38}$$

for $j = 1, \ldots, m$. To evaluate the last term on the right-hand-side we use the Dyson formula

$$e^{tL} = e^{tQL} + \int_0^t e^{(t-s)L} PLe^{sQL} ds \,. \tag{39}$$

This identity is satisfied at $t = 0$ and it can be made plausible for $t > 0$ along the following line. The differentiated form of (39), i.e.

$$\begin{aligned} Le^{tL} &= QLe^{tQL} + PLe^{tQL} + L\left\{e^{tL} - e^{tQL}\right\} \\ &= Le^{tL} + ILe^{tQL} - Le^{tQL} = Le^{tL} \,, \end{aligned} \tag{40}$$

leads to an identity (since $P+Q = I$) for all t. The integrated form (39) will be used now in order to rewrite (38) as

$$\frac{\partial}{\partial t} \varphi_j(t; x) = \mathcal{R}_j(\hat{\varphi}(t; x)) + \int_0^t e^{(t-s)L} K_j(s; \hat{x}) ds + F_j(t; x) \,, \tag{41}$$

where

$$\begin{aligned} F_j(t; x) &= e^{tQL} QLx_j \,, & (42) \\ K_j(t; \hat{x}) &= PLF_j(t, x) \,. & (43) \end{aligned}$$

The first term on the right-hand-side of (41) is the Markovian term, the second is a memory term (since it depends on the whole time history), and the last one is the so called noise term which is statistically dependent on the initial distribution. Equation (41) is not yet our final equation since it contains the initial values x_j for $j = m+1, \ldots, n$. In order to get rid of the latter we multiply by P from the left and introduce the variables

$$\hat{\phi}_j = \hat{\phi}_j(t; \hat{x}) \equiv P\hat{\varphi}_j = P\hat{\varphi}_j(t; x) \tag{44}$$

with

$$\hat{\phi}_j(t = 0, \hat{x}) = \hat{x}_j \quad \text{for } j = 1, \ldots, m \,. \tag{45}$$

Usually, one makes the additional approximation

$$P\mathcal{R}_j(\hat{\varphi}(t; x)) \approx \mathcal{R}_j(P\hat{\varphi}(t, x)) \equiv \mathcal{R}_j(\hat{\phi}(t; \hat{x})) \,. \tag{46}$$

Actually, the latter approximation is not so bad as it might look at the first glance. We have already obtained a decoupled subset of equations for $j = 1, \ldots, m$. In addition, $\hat{\varphi}_j(t = 0; x)) = \hat{\phi}_j(t = 0; \hat{x})) = x_j$ is satisfied. The noise term vanishes after projection since from $P^2 = P$

$$PQ = 0 \tag{47}$$

follows. Putting things together we arrive at

$$\frac{\partial}{\partial t} \hat{\phi}_j(t; \hat{x}) = \mathcal{R}_j[\hat{\phi}(t; x)] + \int_0^t PK_j(t - s; \hat{x}) ds \,. \tag{48}$$

II.2 First results for the simple example

Before continuing with the theoretical background let us come back to the example (6)–(9). We rewrite it in the new notation as

$$\frac{d\varphi_1}{dt} = R_1(\varphi) \equiv \varphi_2 \;, \quad (49)$$

$$\frac{d\varphi_2}{dt} = R_2(\varphi) \equiv -\varphi_1(1+\varphi_3^2) \;, \quad (50)$$

$$\frac{d\varphi_3}{dt} = R_3(\varphi) \equiv \varphi_4 \;, \quad (51)$$

$$\frac{d\varphi_4}{dt} = R_4(\varphi) \equiv -\varphi_3(1+\varphi_1^2) \;. \quad (52)$$

Now the operator $L(x)$ is

$$L(x) = x_2\,\frac{\partial}{\partial x_1} - x_1(1+x_3^2)\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} - x_3(1+x_1^2)\frac{\partial}{\partial x_4}. \quad (53)$$

To calculate $\mathcal{R}_j(\hat{x})$ as defined by (36), we evaluate

$$PLx_1 = x_2 \;, \quad (54)$$

$$PLx_2 = P\left[-x_1(1+x_3^2)\right] = -x_1\left(1+\frac{1}{1+x_1^2}\right) \quad (55)$$

That leads to

$$e^{tL}PLx_1 = e^{tL}x_2 = \varphi_2(t;x) \;, \quad (56)$$

$$e^{tL}PLx_2 = -\varphi_1(t;x)\left[1+\frac{1}{\varphi_1^2(t;x)}\right] \;. \quad (57)$$

After the final projection we obtain the equations

$$\frac{\partial}{\partial t}\,\phi_1(t;\hat{x}) = \phi_2(t;\hat{x}) \;, \quad (58)$$

$$\frac{\partial}{\partial t}\,\phi_2(t;\hat{x}) = -\phi_1(t;\hat{x}) - \frac{\phi_1(t;\hat{x})}{1+\phi_1^2(t;\hat{x})} \;. \quad (59)$$

They are (still) of Hamiltonian form, with

$$\mathcal{H} = \frac{1}{2}(\phi_1^2+\phi_2^2) + \ln\sqrt{1+\phi_1^2} \;. \quad (60)$$

When we compare a typical solution to Eqs. (58) and (59) with the solution of the (trivially) truncated system (11) and (12) we recognize that both solutions show regular oscillations in time. The effect of the additional terms in optimal predictions is a frequency shift. Although we find an effect of the correction term, the optimal prediction solution is not in good agreement with the averaged solution. The latter is shown in Fig. 1. The averaged solution has been obtained by solving Eqs. (6)-(9) for fixed initial values of the first two variables x_1 and x_2, while the initial values for x_3 and x_4 are chosen randomly, and the averaging is over these unresolved initial values with the canonical distribution.

II.3 Non-Markovian contributions

The memory term on the right-hand-side of Eq. (48) is needed to improve the long-time agreement of the approximate solution with the averaged one. There are various strategies to incorporate the memory term. Here we present only the simplest procedures. Let us approximate (i.e. expand around $s=0$)

$$\int_0^t K_j\left[\hat{y}(t-s,x),s\right]ds \approx \int_0^t e^{tL}K_j(\hat{x},0)ds \;, \quad (61)$$

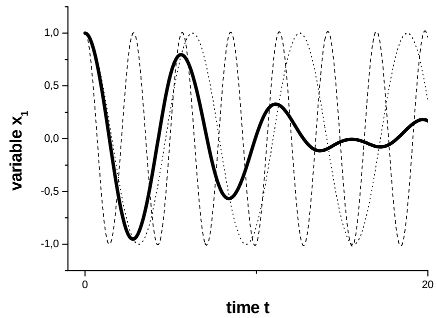

Figure 1: *The broken lines show two different exact solutions of (6)-(9) for two different initial values of the unresolved variables. The solid line is the result of averaging many exact solutions, when each exact solution gets a weight corresponding to the canonical distribution.*

where

$$e^{tL}K_j(\hat{x},0) \equiv K_j\left[\hat{\varphi}(t,x),0\right] = e^{tL}PLQLx_j \;. \quad (62)$$

Let us evaluate the right-hand-side of (62) for $j=1$ and $j=2$ in the case of our example. For $j=1$ we have

$$PLQLx_1 = PLQx_2 = PLx_2 - PLx_2 = 0 \;. \quad (63)$$

On the other hand, for $j=2$

$$PLQLx_2 = -2\frac{x_1^2x_2}{(1+x_1^2)^2} \;. \quad (64)$$

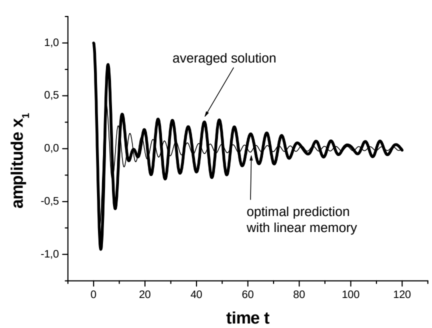

Figure 2: *Comparison of the averaged solution with the optimal prediction including the memory term with a linear t-dependence.*

The simplest procedure is to estimate the integrals as

$$\int_0^t PK_1(t-s;\hat{x})ds = 0 \;, \quad (65)$$

$$\int_0^t PK_2(t-s;\hat{x})ds \approx -2\,\frac{\phi_1^2\phi_2}{(1+\phi_1^2)^2}\,t \;. \quad (66)$$

This memory approximation with a linear t-dependence leads to the improved set of equations

$$\frac{\partial \phi_1}{\partial t} = \phi_2 \, , \tag{67}$$

$$\frac{\partial \phi_2}{\partial t} = -\phi_1 - \frac{\phi_1}{1+\phi_1^2} - 2t \, \frac{\phi_1^2 \phi_2}{(1+\phi_1^2)^2} \, . \tag{68}$$

The solutions of these equations agree much better with the averaged solution. That is shown in Fig. 2. However, for long times the amplitudes become too small because of an overestimate of the memory term.

II.4 Relation to center manifold reduction

As we have seen, the memory term introduces some effective damping due to the transfer of energy to the other modes. However, as we can recognize from Fig. 2 the damping is overestimated when the linear t-approximation is used. That is quite obvious from the explicit form of the memory term which includes phase mixing.

To show how a better approximation might be obtained, we now consider a simple dissipative example, i.e.

$$\dot{x} = -xy \, , \tag{69}$$

$$\dot{y} = -y + x^2 \, . \tag{70}$$

We have a marginally (stable) system. It is expected that in the unstable regime the variable y will be slaved to x. generally, in the neighborhood of an equilibrium, a systematic reduction can be performed. One starts with the eigenvalue spectrum λ of the linearized system. In the linear evolution $\sim \exp(\lambda t)$, eigenvalues with $\Re\lambda < 0$ ($\Re\lambda > 0$) are called stable (unstable), whereas those with $\Re\lambda = 0$ are called central. In the neighborhood of an equilibrium point P of a dynamical system, in general three different types of invariant manifolds exist: The trajectories belonging to the stable manifold M^s are being attracted by P, whereas those of the unstable manifold M^u are being repelled. The dynamics on the center manifold M^c depends on the nonlinearities. For the linearized problem, $E^s \equiv M^s$, $E^u \equiv M^u$, and $E^c \equiv M^c$ are uniquely determined linear subspaces which span the whole space. The transition to the nonlinear system causes (only) deformations of the linearly determined manifolds E^s, E^u, and E^c to M^s, M^u, and M^c. However, the form of the latter crucially depends on the nonlinear terms.

Let us elucidate that behavior for the very simple system introduced above. The equilibrium point P is $(0,0)$, and the linearized system turns out to be

$$\dot{x} = 0 \, , \quad \dot{y} = -y \, . \tag{71}$$

The stable manifold E^s is identical with the y-axis and the center manifold E^c is identical with the x-axis. The linearized problem can be visualized by the graph shown in Fig. 3. Both manifolds will be deformed in the transition to the nonlinear system. The (nonlinear, perturbed) center manifold can be described in the present example by

$$M^c : y = h(x) \tag{72}$$

with $h(0) = h'(0) = 0$. Using that ansatz for the center manifold of (69) and (70) we obtain

$$\dot{x} = -x \, h(x) \, . \tag{73}$$

Differentiating (72) with respect to t, leads to

$$\dot{y} = h'(x) \, \dot{x} \tag{74}$$

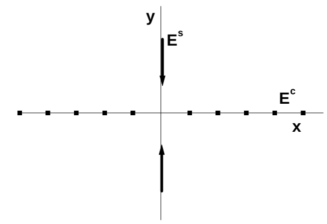

Figure 3: *The linear stable (E^s) and center manifold (E^c) for the example (69) and (70)*

or

$$-h(x) + x^2 = h'(x)[-xh(x)] \, , \tag{75}$$

i.e. a differential equation for $h = h(x)$. Making a power series ansatz $h(x) = cx^2 + dx^3 + \cdots$ we find $c = 1$. The (nonlinear) center manifold is thus given by

$$y = x^2 + \cdots \, , \tag{76}$$

and the dynamics on it follows from

$$\dot{x} = -x^3 + \cdots \, , \tag{77}$$

i.e. the trajectories are being attracted by P. For an illustration see Fig. 4. For the present (simple) example it

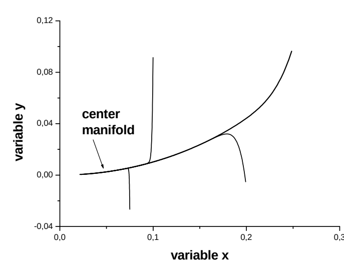

Figure 4: *Solution of the simple model (69) and (70) for four different initial conditions. For small x and y always $y \approx x^2$ holds.*

is possible to straightforwardly compare the exact solution with the center manifold reduction. First, let us formally solve the equation for y by the variation of constant method. The result is

$$y = y_0 e^{-t} + e^{-t} \int_0^t e^s \, [x(s)]^2 \, ds \, . \tag{78}$$

Here, y_0 is the initial value $y_0 = y(t=0)$. Obviously, at larger times the initial value will be "forgotten", and in the asymptotic regime we may use only the second term on the right-hand-side. Having in mind the smallness of x, we may expand (by integration by parts)

$$e^{-t} \int_0^t e^s x^2 ds = e^{-t} \left[e^s x^2 \right]_{s=0}^{s=t} - e^{-t} \int_0^t ds e^s \, \frac{d}{ds} \, x^2$$

$$\approx [x(t)]^2 + \text{"higher order terms"} \, . \tag{79}$$

Thus, with

$$y \approx x^2$$

we directly obtain the center manifold equation

$$\dot{x} = -x^3 + \text{"higher order terms"} . \tag{80}$$

Now we derive that result from the projection method. In the present notation we define

$$L = -xy \frac{\partial}{\partial x} + (-y + x^2) \frac{\partial}{\partial y} \tag{81}$$

and evaluate the memory term

$$\int_0^t e^{(t-s)L} K_j(\hat{x}, s) ds = e^{tL} \int_0^t e^{-sL} P L e^{sQL} Q L x_j . \tag{82}$$

In the following we write

$$Px = x \ , \quad Py = \langle y \rangle \ , \quad Py^2 = \left\langle y^2 \right\rangle \ , \ldots$$

and do not specify the detailed projection procedure. For reason of demonstration we shall assume during the calculation that $\langle y \rangle , \left\langle y^2 \right\rangle , \ldots$ do not become functions of x. In the final application we may use a narrow distribution with $\langle y \rangle = y_0$, $\left\langle y^2 \right\rangle = y_0^2$. Straightforward calculations lead to

$$QLx \equiv (1 - P)Lx = -(1 - P)xy = -xy + x \langle y \rangle \ , \tag{83}$$

$$\begin{aligned} PL\left[-xy + x\langle y \rangle\right] &= Pxy^2 - P(-y + x^2)x + P(-xy\langle y \rangle) \\ &= x\left\langle y^2 \right\rangle + x\langle y \rangle - x^3 - x\langle y \rangle^2 \\ &= x\left[\left\langle y^2 \right\rangle - \langle y \rangle \langle y \rangle\right] + x\langle y \rangle - x^3 \, . \end{aligned} \tag{84}$$

The latter formula will be useful in the evaluation of the integrand of (82) for $s = 0$. Looking at the right-hand-side of (82), we recognize two important factors. First, the operator $\exp(tL)$ will perform the transformation $x(t = 0) \to x(t)$. Secondly, the factor $\exp(-sL)$ is responsible for a rapid drop of the integrand from its value at $s = 0$. [Let us assume the characteristic decay time $s_0 \sim \mathcal{O}(1)$].

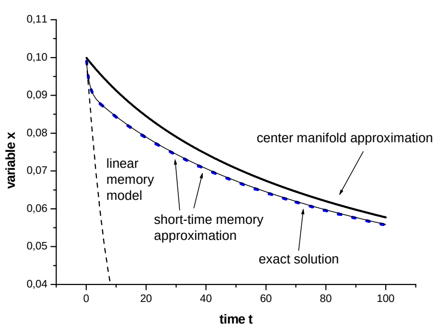

Figure 5: *For the simple example (69) and (70) the optimal prediction (with a short-time memory approximation) is compared with the center manifold reduction.*

To determine the averaged equation we also have to calculate the Markovian term

$$\begin{aligned} \mathcal{R}_1\left(\hat{\varphi}(t;x)\right) &\hat{=} e^{tL} P L x = e^{tL}\left\{-x(0)\langle y \rangle\right\} \\ &= -x(t)\langle y \rangle \ . \end{aligned} \tag{85}$$

Then, for $\left\langle y^2 \right\rangle = \langle y \rangle^2$ the linear t-approximation would lead to the averaged evolution equation of the form

$$\dot{x} \approx -x\left[\langle y \rangle - \langle y \rangle t\right] - x^3 t \ . \tag{86}$$

Fig. 5 shows this solution in comparison to the exact solution for the same value $\langle y \rangle = y_0$. The agreement is only satisfactory for small t. For larger t, the linear extrapolation of the memory term is not valid anymore, in agreement with the observation of a finite memory time. In general, the integrand $e^{-sL} K_j(\hat{x}, s)$ will have a time-dependence as qualitatively been shown in Fig. 6.

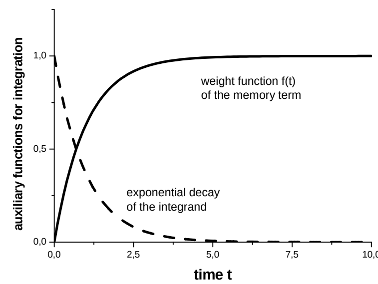

Figure 6: *Qualitative picture of the memory term for the simple model (69) and (70).*

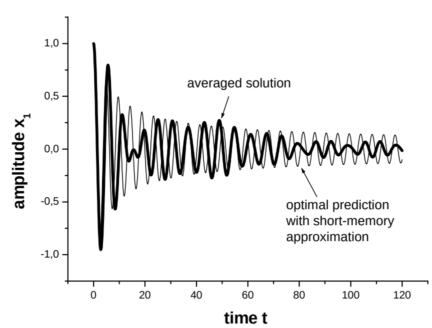

Figure 7: *Comparison of the averaged solution of (6)-(9) with the optimal prediction including the memory term with a short-time memory approximation.*

Choosing all characteristic quantities of order one, instead of the linear growth $\sim t$ we expect a factor $f(t) \approx \left(1 - e^{-t}\right)$. Therefore, [for $\left\langle y^2 \right\rangle = \langle y \rangle \langle y \rangle$] we propose instead of (86) the short-time memory model

$$\dot{x} \approx -\left[1 - f(t)\right] x \langle y \rangle - f(t) x^3 \ , \tag{87}$$

where, e.g.,

$$f(t) \approx 1 - e^{-t} \ . \tag{88}$$

Fig. 5 shows the excellent agreement of this approximation with the exact solution. It is also interesting that in the limit $t \to \infty$ we recover the center manifold result

$$y \approx -x^2 \ . \tag{89}$$

II.5 Back to the simple example

After having learned more about the qualitative and quantitative behavior of the memory term, we apply a similar procedure to the Hamiltonian example (6)-(9). Instead of (67) and (68) we now use

$$\frac{\partial \phi_1}{\partial t} = \phi_2 \,, \tag{90}$$

$$\frac{\partial \phi_2}{\partial t} = -\phi_1 - \frac{\phi_1}{1+\phi_1^2} - 2f(t)\,\frac{\phi_1^2 \phi_2}{(1+\phi_1^2)^2} \,. \tag{91}$$

Again,

$$f(t) \approx 1 - e^{-t} \tag{92}$$

will be used for demonstration. The solution of Eqs. (90) and (91) is shown in Fig. 7 and compared to the averaged solution. As expected, the energy transfer to the other modes is less pronounced. However, the short-time memory approximation seems to work better than the linear t-prediction.

One should have in mind the principal difference in arguments for dissipative and Hamiltonian systems, respectively. In dissipative systems, the decay time can be estimated quite naturally by the linear damping decrement. This explains the excellent agreement in Fig. 5. For Hamiltonian systems, a quite complicated phase-mixing occurs, which is much more difficult to estimate.

III. STATISTICAL PHYSICS FOR GASES AND FLUIDS

After having discussed reduction processes, we now emphasize the probabilistic view. Physical kinetics deals with the one-particle distribution function $f(\vec{r},\vec{v},t)$ for the probability to find a particle at time t at position $\vec{r}$ with velocity $\vec{v}$ irrespective of the actual data of the other particles. We start with the classical consideration by Boltzmann.

III.1 Boltzmann equation with binary collisions

For a collision, we can define two characteristic times: τ_c, the time of a collision, which is related to a finite range r_c of the interaction potential via $\tau_c = r_c/v_{th}$, and the time between two collisions $\tau = 1/nr_c^2 v_{th}$. For a neutral gas, r_c is well-defined, and in the region

$$\tau_c \ll \tau \rightarrow r_c \ll n^{-1/3} \tag{93}$$

we can consider independent binary collisions. Obviously, this situation will not occur in a plasma with long-range-forces. Note that condition (93) means dilute gases. Boltzmann derived an equation for the distribution function of neutral particles. Let us consider volume elements d^3r and d^3v, being large compared to microscopic volumes but small on macroscopic scales. During the Hamiltonian dynamics in the presence of an external force $\vec{F}$ (but without interactions) motion occurs from $d^3r\ d^3v$ to $d^3r'\ d^3v'$ with $d^3r\ d^3v = d^3r'\ d^3v'$. Without collisions (interactions) the particle numbers in the volume elements would be conserved. Collisions will cause changes according to

$$\left[f\left(\vec{r}+\vec{v}dt, \vec{v}+\frac{1}{m}\,\vec{F}dt, t+dt\right) - f(\vec{r},\vec{v},t)\right] d^3r\ d^3v$$
$$= \left.\frac{df}{dt}\right|_{coll} dt\, d^3r\, d^3v. \tag{94}$$

The left-hand-side is simplified by Taylor expansion. The right-hand-side is evaluated via a "Stoßzahl-Ansatz",

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\nabla + \frac{1}{m}\,\vec{F}\cdot\frac{\partial}{\partial \vec{v}}\right] f(\vec{r},\vec{v},t) \tag{95}$$

$$= \int d^3v_2 \int d^3v_3 \int d^3v_4\ W(\vec{v},\vec{v}_2;\vec{v}_3,\vec{v}_4)[f_3 f_4 - f_1 f_2] \,,$$

where $f_i \equiv f(\vec{r},\vec{v}_i,t)$ for $i = 1,2,3,4$ with $\vec{v}_1 = \vec{v}$. $W(\vec{v}_1,\vec{v}_2;\vec{v}_3,\vec{v}_4)$ is the probability that the collision of two particles with velocities $\vec{v}_1$ and $\vec{v}_2$ results in new particle velocities $\vec{v}_3$ and $\vec{v}_4$. Because of obvious symmetry relations for W, the right-hand-side of (96) simply takes care of the particle density balance. The evaluation of W is not trivial. Here, we can already remark that conservation laws require

$$W(\vec{v}_1,\vec{v}_2;\vec{v}_3,\vec{v}_4) \sim \delta\left[\frac{v_3^2+v_4^2}{2} - \frac{v_1^2+v_2^2}{2}\right]$$
$$\times\delta(\vec{v}_3+\vec{v}_4-\vec{v}_1-\vec{v}_2) \,. \tag{96}$$

When binary and central collisions are considered, we can relate the, e.g., loss rate to the collision cross section σ. Consider a particle with velocity $\vec{v}_2$ approaching another particle with velocity $\vec{v}_1$ with a collision parameter s. The incoming (relative) flux is $f(\vec{r},\vec{v}_2,t)|\vec{v}_2-\vec{v}_1|d^3v_2$, and we have the scattering relation

$$f(\vec{r},\vec{v}_2,t)|\vec{v}_2-\vec{v}_1|d^3v_2\,(-s\,d\varphi\,ds)$$
$$= f(\vec{r},\vec{v}_2,t)|\vec{v}_2-\vec{v}_1|d^3v_2\sigma(\Omega,|\vec{v}_1-\vec{v}_2|)d\Omega \,. \tag{97}$$

To include all collisions during time dt with particles of a certain velocity $\vec{v}_1 = \vec{v}$ in d^3v_1 (and volume element d^3r) we have to multiply by $f(\vec{r},\vec{v}_1,t)$ to get the loss rate

$$l\ d^3r\ d^3v_1\ dt \tag{98}$$
$$= \int d^3v_2 \int d\Omega|\vec{v}_2-\vec{v}_1|\sigma(\Omega,|\vec{v}_2-\vec{v}_1|)f_1 f_2\ d^3r\ d^3v_1\ dt$$

where $f_{1,2} \equiv f(\vec{r},\vec{v}_{1,2},t)$.

In a similar way we can calculate the gain rate by particles with velocities $(\vec{v}_1', d^3v_1')$ being scattered into $(\vec{v}_1, d^3v_1)$:

$$g\ d^3r\ d^3v_1\ dt = d^3v_1' \int d\Omega \int d^3v_2'|\vec{v}\,'_1 - \vec{v}\,'_2| \tag{99}$$
$$\times\sigma\left(\Omega,|\vec{v}\,'_1-\vec{v}\,'_2|\right) f_1' f_2'\ d^3r\ dt \,,$$

where $f_{1,2}' = f(\vec{r},\vec{v}\,'_{1,2},t)$, and $\vec{v}\,'_1, \vec{v}\,'_2$ are the two initial velocities which after the collision evolve into $\vec{v}_1$ and $\vec{v}_2$. If $\vec{v}_1$ is fixed, momentum conservation determines $\vec{v}_2$.

Now, $d^3v_1'\ d^3v_2' = d^3v_1\ d^3v_2$ (Jacobi determinant equal to 1), and we may write

$$g\ d^3r\ d^3v_1\ dt = d^3v_1 \int d^3v_2 \int d\Omega|\vec{v}_1-\vec{v}_2|\sigma(\Omega,|\vec{v}_1-\vec{v}_2|)$$
$$\times f_1'\ f_2'\ d^3r\ dt \,. \tag{100}$$

The collision term then follows as

$$\left.\frac{df}{dt}\right|_{coll} \equiv g - l = \int d^3v_2\ d\Omega|\vec{v}_2-\vec{v}_1|\sigma(\Omega,|\vec{v}_2-\vec{v}_1|)$$
$$\times\left[f_1'\,f_2' - f_1\,f_2\right] . \tag{101}$$

A comparison with (96) shows the expected similarity since

$$\int d\Omega|\vec{v}_2-\vec{v}_1| = 4\int d^3v_1' \int d^3v_2'\ \delta\left(\frac{v_1'^2+v_2'^2}{2}\right. \tag{102}$$
$$\left.-\frac{v_1^2+v_2^2}{2}\right)\delta\left(\vec{v}\,'_1+\vec{v}\,'_2-\vec{v}_1-\vec{v}_2\right)$$

can be verified by straightforward calculations

The Boltzmann equation is a non-time-symmetric equation, i.e. it shows irreversibility. The latter has been demonstrated by Boltzmann already when deriving the so called H-theorem. Consider the functional

$$H(\vec{r},t) = \int d^3v\, f(\vec{r},\vec{v},t)\, \log f(\vec{r},\vec{v},t) \tag{103}$$

and its time-derivative $\dot{H} = dH/dt$. Although more general conclusions are available, let us consider the case when no external field $\vec{F}$ is present. Then it is straightforward to show

$$\dot{H} \leq 0 \,. \tag{104}$$

Since $f \log f$ is bounded from below, the distribution function tends to the minimum of H which is realized for a Maxwellian.

III.2 Hydrodynamic equations

The Boltzmann equation is a closed integro-differential equation for the one-particle distribution function f, where here and in the following – whenever appropriate – the index points at the variables, i.e. $f_1 = f(\vec{r}_1,\vec{v}_1,t)$. The left-hand-side of Eq. (94) is the free streaming term, whereas the right-hand-side is the binary collision term. The purpose of plasma kinetic theory is to derive the corresponding equation for a system of charged particles. Before presenting the latter let us demonstrate here already how the hydrodynamic description is linked to kinetic theory.

Averaging over velocity space,

$$\langle \cdots \rangle = \frac{\int d^3v \cdots f}{\int d^3v\, f} \,, \tag{105}$$

we can introduce the hydrodynamic variables (particle density, mean velocity, temperature, respectively)

$$n\,(\vec{r},t) = \int d^3v\, f \,,\ \vec{u} := \langle \vec{v} \rangle \,,\ T := \frac{1}{3} m \langle |\vec{v}-\vec{u}|^2 \rangle \,. \tag{106}$$

By straightforward integrations we obtain the following generally valid equations:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0 \,, \tag{107}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} = \frac{1}{m}\vec{F} - \frac{1}{mn}\nabla\cdot\underline{\underline{P}} \,, \tag{108}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right) T = -\frac{2}{3n}\nabla\cdot\vec{q} - \frac{2}{3mn}\underline{\underline{P}} : \underline{\underline{\Lambda}} \,, \tag{109}$$

where

$$P_{ij} := mn\langle (v_i - u_i)(v_j - u_j)\rangle \,, \tag{110}$$

$$\vec{q} := \frac{1}{2} mn \langle (\vec{v}-\vec{u})|\vec{v}-\vec{u}|^2\rangle \,, \tag{111}$$

$$\Lambda_{ij} := \frac{1}{2} m \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{112}$$

are higher moments. Note that we have not yet obtained the hydrodynamic equations, since the higher moments are not known. The evaluation of, e.g., the pressure tensor and the heat current, is the highly non-trivial task of transport theory. The latter uses quite effectively the kinetic theory, as will be outlined now for our introductory example of a neutral fluid.

Let us abbreviate the kinetic equation by

$$\frac{\partial f}{\partial t} + Df = \frac{1}{\epsilon} J(f|f) \,. \tag{113}$$

Here a very important assumption has been introduced. We assume the (neutral) system to be collision-dominated, by introducing the (huge but only symbolic) factor $1/\epsilon$ in front of the collision term. Having an equation of that type, the multiple-scale analysis is the appropriate method of solution. We expand

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots \,, \tag{114}$$

$$f^{(\nu)} = f^{(\nu)}(\vec{r},\vec{v};t_0,t_1,t_2,\ldots) \,, \tag{115}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2} + \cdots \,, \tag{116}$$

to get to lowest order

$$J^{(0)}\left(f^{(0)}|f^{(0)}\right) = 0 \,. \tag{117}$$

Its solution is the celebrated Maxwell distribution

$$f^{(0)}(\vec{r},\vec{v};t) = n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left[-\frac{m}{2T}(\vec{v}-\vec{u})^2\right] \;; \tag{118}$$

the hydrodynamic variables appear as parameters. (It is important to note that the parameters appearing in the lowest order distribution function are by definition the exact hydrodynamic quantities.) With the help of the Maxwell distribution we can evaluate the higher moments. Introducing the results into the general equations for the first three moments leads to the famous Euler equations

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0 \,, \tag{119}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} = \frac{\vec{F}}{m} - \frac{1}{mn}\nabla p \,, \tag{120}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right) T = -\frac{2}{3} T\nabla\cdot\vec{u} \,. \tag{121}$$

A better description is obtained when higher order corrections are included. The equation for $f^{(1)}$ is

$$\left(\frac{\partial}{\partial t_0} + D\right) f^{(0)} = J^{(1)}\left(f^{(0)}|f^{(1)}\right) \,, \tag{122}$$

with the constraint

$$\int d^3v f^{(1)} \begin{pmatrix} 1 \\ \vec{v} \\ v^2 \end{pmatrix} = 0 \,. \tag{123}$$

Its solution is a quite sophisticated task. We write in an abbreviated form

$$f^{(1)} = -\left[\frac{1}{T}\frac{\partial T}{\partial x_i} g_i \mathcal{F}(g) + \frac{1}{T}\Lambda_{ij}\left(g_i g_j - \frac{1}{3}\delta_{ij} g^2\right)\mathcal{G}(g)\right] f^{(0)} \,, \tag{124}$$

where $\vec{g} := \vec{v} - \vec{u}$. The functions $\mathcal{F}$ and $\mathcal{G}$ can be found by, e.g., series solutions. Incorporating the first order, the basic equations are the Navier-Stokes equations

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}) = 0 \,, \tag{125}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} = \frac{\vec{F}}{m} - \frac{1}{mn}\nabla\left(p - \frac{\mu}{3}\nabla\cdot\vec{u}\right) + \frac{\mu}{mn}\nabla^2\vec{u} \,, \tag{126}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)T = -\frac{2}{3}\left(\nabla\cdot\vec{u}\right)T + \frac{2K}{3n}\nabla^2 T \,. \tag{127}$$

The pressure tensor and the heat current are in this approximation

$$P_{ij} = p\delta_{ij} - \frac{2\mu}{m}\left(\Lambda_{ij} - \frac{m}{3}\delta_{ij}\nabla\cdot\vec{u}\right) \,, \tag{128}$$

$$\vec{q} = -K\,\nabla T \,, \tag{129}$$

where the viscosity and heat conductivity are calculated as

$$\mu = \frac{m^2}{15T}\int d^3g\, g^4\, f^{(0)}\mathcal{G}(g) \,, \tag{130}$$

$$K = \frac{m}{6T}\int d^3g\, g^4\, f^{(0)}\mathcal{F}(g) \,, \tag{131}$$

respectively.

III.3 The next steps for plasmas

What can we learn from this case? First of all, we have to develop the proper kinetic equation for a system of charged particles. That equation should be in the form of a closed Markovian description for the one-particle distribution function(s). With that equation we can play around, i.e. determine, e.g., the dispersive properties, instabilities, *etc* of the system. But we should have in mind that the distribution function has three space coordinates, the three velocity components, and the time as arguments. To find some solutions beyond linearization is very hard. The idea is to reduce the number of arguments whenever possible. For fast gyrations, it will be advisable to get rid of the gyro-phase (by integrating over the gyro-phase, called gyro-phase averaging) and to end up with some drift-kinetic or gyro-kinetic description. One might even be more rigorous and switch over to a macroscopic description by integrating over the velocity space. This will generate a hierarchy problem which should be truncated by some appropriate solution of the kinetic equation. A consistent truncation procedure is known as transport theory. We have seen an elementary form of transport theory already when discussing the Navier-Stokes equations, with results for the pressure tensor and the heat conductivity. But we should have in mind that any simplification goes along with a loss of information. For example, when using the macroscopic description, we have to restrict ourselves to hydrodynamic length and time scales. Let us begin now with the first steps of such an ambiguous program for plasmas. More advanced problems will be considered in other contributions to this volume.

IV. CLASSICAL KINETIC PLASMA EQUATIONS

The long range nature of the Coulomb interactions introduces a qualitatively new aspect when plasmas are considered. In the previous section we have seen that close individual collisions introduce a stochastic behavior which leads to irreversibility. As soon as many particles contribute to the electric field at the position of a particle, the stochastic nature of the field becomes less pronounced. The electric field generated by a huge number of particles gets a macroscopic character. It becomes responsible for the collective behavior of plasmas. The stochastic nature of collisions is only guaranteed for close collisions caused by one (or few particles). When λ_D stands for the range of a screened Coulomb potential, then $n\lambda_D^3 \gg 1$ will mean that collective effects are important (or dominating). However, as long as they do not show a stochastic nature (then we call them turbulent fields) they do not contribute to transport. Irreversibility and transport will be then caused only by strong collisions of individual particles.

IV.1 General outline of the derivation

Starting from the Liouville equation we want to derive a kinetic plasma equation, i.e. a closed equation for the one-particle distribution function

$$f^\alpha(\vec{q},\vec{p};t) \equiv f_1^\alpha(\vec{q}_1,\vec{p}_1;t) := N_\alpha \int d^3q_2 \ldots d^3q_N d^3p_2 \ldots d^3p_N\, \rho, \tag{132}$$

where α designates the species of the particle under consideration. This (normalized) function describes the probability of finding a particle of species α at position $\vec{q}$ with momentum $\vec{p}$ at time t. In a similar way one can define multiple-particle distribution functions, e.g. the two-particle distributions

$$f_2^{\alpha\alpha} = N_\alpha(N_\alpha - 1)\int d^3q_3 \ldots d^3q_N d^3p_3 \ldots d^3p_N \rho, \tag{133}$$

$$f_2^{\alpha\beta} = N_\alpha N_\beta \int d^3q_3 \ldots d^3q_N d^3p_3 \ldots d^3p_N\, \rho. \tag{134}$$

The generalizations are obvious. In the absence of magnetic fields, the change from $\vec{p} = m\vec{v}$ to $\vec{v}$ as a variable is trivial. The idea is to obtain the kinetic equation for the one-particle distribution function by integrating the Liouville equation over the irrelevant coordinates and momenta. And again the procedure is not trivial since we get a hierarchy of coupled equations. Let us elucidate this important point a little bit more in detail.

Introducing the Liouville operator

$$L = -\sum_{j=1}^{N}\left[\vec{v}_j\cdot\frac{\partial}{\partial\vec{q}_j} - \frac{e_j}{m_j}\frac{\partial\phi}{\partial\vec{q}_j}\cdot\frac{\partial}{\partial\vec{v}_j}\right] + \sum_{j=1}^{N}\sum_{i=1}^{j-1}\frac{1}{m_j}\frac{\partial\phi_{ij}}{\partial\vec{q}_j}\cdot\frac{\partial}{\partial\vec{v}_j} \equiv L^{(1)} + L^{(2)} \tag{135}$$

for particles in an external potential ϕ and with interaction potential

$$\phi_{ij} = \frac{e_i e_j}{|\vec{q}_i - \vec{q}_j|} \,, \tag{136}$$

we get a one-particle propagator $L^{(1)}$ and an interaction contribution $L^{(2)}$. The latter depends on coordinates of two particles in a non-separable manner. It causes the main problems. Note that the Liouville equation can be written in the form

$$\frac{\partial\rho}{\partial t} = L\rho. \tag{137}$$

Now integrating (137) over all the coordinates and momenta of the other particles, except $\vec{q}_1$ and $\vec{p}_1$ of the particle under consideration, we obtain after some straightforward manipulations

$$\partial_t f^\alpha(\vec{q}_1, \vec{v}_1; t) = L_1^\alpha\, f^\alpha(\vec{q}_1, \vec{v}_1; t)$$

$$+ \sum_{\beta=e,i} \int d^3q_2\, d^3v_2\, L_{12}^{\alpha\beta}\, f^{\alpha\beta}(\vec{q}_1, \vec{v}_1, \vec{q}_2, \vec{v}_2; t)\ , \quad (138)$$

where $\partial_t = \partial/\partial t$. Here and in the following we omit some indices when no confusion is expected. Also, the L-operators follow from (135) in a straightforward manner. Obviously, because of the inter-particle interactions, this equation for f^α contains the two-particle distribution function $f^{\alpha\beta}$. The latter we split into two parts:

$$f^{\alpha\beta}(\vec{q}_1, \vec{v}_1, \vec{q}_2, \vec{v}_2; t) \;=\; f^\alpha(\vec{q}_1, \vec{v}_1; t) f^\beta(\vec{q}_2, \vec{v}_2; t)$$

$$+ g^{\alpha\beta}(\vec{q}_1, \vec{v}_1, \vec{q}_2, \vec{v}_2; t), \quad (139)$$

where the first contribution on the right-hand-side is the dominating one in dilute gases, when particles approximately move independently, and the second contribution measures the correlation. In a similar way, we can define the triple correlation function via

$$f^{\alpha\beta\gamma} = f^\alpha f^\beta f^\gamma + f^\alpha g^{\beta\gamma} + f^\beta g^{\alpha\gamma} + f^\gamma g^{\alpha\beta} + g^{\alpha\beta\gamma}. \quad (140)$$

In all following discussions we shall assume $g^{\alpha\beta\gamma} \approx 0$, meaning that close clusters of three particles are very rare. This assumption is consistent with previous considerations since $n\lambda_D^3 \gg 1$ is good for dilute systems. By the assumption $g^{\alpha\beta\gamma} \approx 0$ we close the BBGKY (Bogoliubov, Born, Green, Kirkwood, Yvon) hierarchy which expresses the fact that the equation for f_1 contains f_2, the equation for f_2 contains f_3, and so on. But still we have not succeeded in a kinetic equation. By a kinetic equation we mean a closed equation for the one-particle distribution function. Our present state of calculation has produced the following coupled set of equations

$$\partial_t f^\alpha(\vec{q}_1, \vec{v}_1; t) = L_1^\alpha f^\alpha(\vec{q}_1, \vec{v}_1; t)$$

$$+ \sum_{\beta=e,i} \int d^6 2\, L_{12}^{\alpha\beta}\, f^\alpha(\vec{q}_1, \vec{v}_1; t) f^\beta(\vec{q}_2, \vec{v}_2; t)$$

$$+ \sum_{\beta=e,i} \int d^6 2\, L_{12}^{\alpha\beta} g^{\alpha\beta}(\vec{q}_1, \vec{v}_1, \vec{q}_2, \vec{v}_2; t), \quad (141)$$

$$\partial_t g^{\alpha\beta}(\vec{q}_1, \vec{v}_1, \vec{q}_2, \vec{v}_2; t) = (L_1^\alpha + L_2^\beta) g^{\alpha\beta}(\vec{q}_1, \vec{v}_1, \vec{q}_2, \vec{v}_2; t)$$

$$+ \quad L_{12}^{\alpha\beta} g^{\alpha\beta}(\vec{q}_1, \vec{v}_1, \vec{q}_2, \vec{v}_2; t)$$

$$+ \quad \sum_{\gamma=e,i} \int d^6 3 \left[L_{13}^{\alpha\gamma} f^\alpha(\vec{q}_1, \vec{v}_1; t) g^{\beta\gamma}(\vec{q}_2, \vec{v}_2, \vec{q}_3, \vec{v}_3; t) \right.$$

$$+ \quad L_{23}^{\beta\gamma} f^\beta(\vec{q}_2, \vec{v}_2; t) g^{\alpha\gamma}(\vec{q}_1, \vec{v}_1, \vec{q}_3, \vec{v}_3; t)$$

$$+ \quad \left. (L_{13}^{\alpha\gamma} + L_{23}^{\beta\gamma}) f^\gamma(\vec{q}_3, \vec{v}_3; t) g^{\alpha\beta}(\vec{q}_1, \vec{v}_1, \vec{q}_2, \vec{v}_2; t) \right]$$

$$+ \quad L_{12}^{\alpha\beta} f^\alpha(\vec{q}_1, \vec{v}_1; t) f^\beta(\vec{q}_2, \vec{v}_2; t). \quad (142)$$

Note that we have introduced the symbol $d^6 2$ to indicate integration over position and velocity of particle 2. We have to eliminate $g^{\alpha\beta}$ in order to get one closed equation for f^α. Besides mathematical also physical problems arise if we proceed in a straightforward manner (if there exists any). Suppose we would be able to solve (142) for $g^{\alpha\beta}$. Then the correlation function would depend on the whole time-history (which in complete detail is actually not relevant). Kinetic regime means that the correlation function depends on its variables only functionally through the one-particle distribution functions; formally

$$g^{\alpha\beta}(t) \approx g^{\alpha\beta}[f(t)]. \quad (143)$$

Now let us discuss which approximations can be applied to the system (141) and (142). The first approximation is due to Vlasov: We close equation (141) directly by putting $g^{\alpha\beta} = 0$. In the second approximation, due to Landau, we neglect in (142) for small coupling ($g^{\alpha\beta} \ll f^\alpha f^\beta$) all terms which contain contributions of particle 3. Then (142) simplifies to

$$\left[\partial_t - L_1^\alpha - L_2^\beta\right] g^{\alpha\beta} = L_{12}^{\alpha\beta}\, f^\alpha(\vec{q}_1, \vec{v}_1; t) f^\beta(\vec{q}_2, \vec{v}_2; t). \quad (144)$$

The third and most advanced ansatz is due to Balescu, Lenard, and Guernsey. Screening contributions of the third particle are taken into account when (142) is approximated by

$$\left[\partial_t - L_1^\alpha - L_2^\beta\right] g^{\alpha\beta} = L_{12}^{\alpha\beta}\, f^\alpha(\vec{q}_1, \vec{v}_1; t) f^\beta(\vec{q}_2, \vec{v}_2; t)$$

$$+ \quad \sum_{\gamma=e,i} \int d^6 3 \left[L_{13}^{\alpha\gamma} f^\alpha(\vec{q}_1, \vec{v}_1; t) g^{\beta\gamma}(\vec{q}_2, \vec{v}_2, \vec{q}_3, \vec{v}_3; t) \right.$$

$$+ \quad \left. L_{23}^{\beta\gamma}\, f^\beta(\vec{q}_2, \vec{v}_2; t) g^{\alpha\gamma}(\vec{q}_1, \vec{v}_1, \vec{q}_3, \vec{v}_3; t) \right]. \quad (145)$$

Note that in all cases we shall arrive at a kinetic equation of the form

$$\partial_t f^\alpha(\vec{q}_1, \vec{v}_1; t) = L_1^\alpha f^\alpha(\vec{q}_1, \vec{v}_1; t)$$

$$+ \quad \sum_{\beta=e,i} \int d^3q_2 d^3v_2 L_{12}^{\alpha\beta}\, f^\alpha(\vec{q}_1, \vec{v}_1; t) f^\beta(\vec{q}_2, \vec{v}_2; t)$$

$$+ \quad K^\alpha\{f^\alpha(t)\}\ . \quad (146)$$

IV.2 The Vlasov equation

In the Vlasov approach, we have $K^\alpha \equiv 0$, and (141) can be rewritten in the following form:

$$\frac{\partial f^\alpha}{\partial t} \quad + \quad \vec{v}_1 \cdot \frac{\partial f^\alpha}{\partial \vec{q}_1} + \frac{e_\alpha}{m_\alpha} \left[\frac{1}{c} \vec{v}_1 \times \vec{B}(\vec{q}_1) + \vec{E}_0 \right.$$

$$+ \quad \left. \vec{E}(\vec{q}_1; t) \right] \cdot \frac{\partial f^\alpha}{\partial \vec{v}_1} = 0, \quad (147)$$

where in the electrostatic approximation ($\nabla \times \vec{E} = 0$, external $\vec{B}$ and external $\vec{E}_0$) the self-consistent electric field $\vec{E}$ follows from Poisson's equation

$$\nabla \cdot \vec{E} = 4\pi \sum_{\beta=e,i} e_\beta \int d^3v_1 f^\beta(\vec{q}_1, \vec{v}_1; t). \quad (148)$$

The Vlasov equation takes care of the collective effects and is exactly valid in the limit $n\lambda_D^3 \to \infty$. Because of the long range nature of the interaction potential, particles move under the action of the electric field produced by the others; strong binary reflections are small ($\to 0$). The Vlasov equation expresses the fact that in the limit $n\lambda_D^3 \to \infty$ the many small influences of all the other particles can dominate over the rare strong deflections (due to close interactions).

IV.3 The Landau-Fokker-Planck equation

When solving (144) for the Landau-Fokker-Planck equation we have to remember our physical implication (143). One can estimate that over times of the order

$$\tau_c = \max(\omega_{pe}^{-1}, \omega_{pi}^{-1}) \; , \tag{149}$$

i.e. the mean time for a collision process, the initial correlations disappear. Here, the plasma frequencies

$$\omega_{p\nu} = v_{t\nu}/\lambda_{D\nu} \tag{150}$$

have been defined. In kinetic theory we are not interested in the relaxation phenomena on an atomic scale. In this brief summary we cannot present the details of the algebra; instead we summarize the result for the non-vanishing collision integral

$$\begin{aligned} K^\alpha &= \sum_{\beta=e,i} 2\pi e_\alpha^2 e_\beta^2 \ln \Lambda \int d^3 v_2 \frac{1}{m_\alpha} \frac{\partial}{\partial v_{1\nu}} G_{\nu\mu}(\vec{g}) \qquad (151) \\ &\times \left[\frac{1}{m_\alpha} \frac{\partial}{\partial v_{1\mu}} - \frac{1}{m_\beta} \frac{\partial}{\partial v_{2\mu}} \right] f^\alpha(\vec{q}_1, \vec{v}_1; t) f^\beta(\vec{q}_1, \vec{v}_2; t) \; , \end{aligned}$$

where

$$G_{\nu\mu}(\vec{g}) = \frac{g^2 \delta_{\nu\mu} - g_\nu g_\mu}{g^3} \tag{152}$$

is the Landau tensor. Several comments are in order. First, we have used $\vec{g} = \vec{v}_2 - \vec{v}_1$ and have introduced $\ln \Lambda$ as the average Coulomb logarithm,

$$\ln \Lambda = \ln \frac{3\lambda_D (T_e + T_i)}{2e^2}. \tag{153}$$

Secondly, during the algebraic manipulations the divergent integral

$$A_{\alpha\beta}(0, \infty) := 2\pi e_\alpha^2 e_\beta^2 \int_0^\infty dk \frac{1}{k} \tag{154}$$

appears. The divergence for $k \to \infty$ originates from small distances in the Coulomb potential when the weak coupling approximation fails anyhow. We have used

$$k_{\max} \approx 3T_\alpha / e_\alpha^2 \; , \tag{155}$$

since $e_\alpha^2/3T_\alpha$ is the collision parameter for 90° deflections. Finally, the divergence for $k \to 0$, corresponding to large distances in the Coulomb potential, originates from the fact that the shielding is not taken appropriately. We can either introduce a cut-off at $k_{\min} \approx 1/\lambda_D$ or replace in the evaluation of the corresponding integrals the Coulomb potential by a (static) Debye potential in an ad hoc manner. Balescu and Lenard have solved the latter problem in a mathematically more rigorous manner.

IV.4 The Balescu-Lenard equation

Using the form (145), a mathematically rather sophisticated and physically profound procedure leads to the collision operator

$$K = \frac{1}{m_e^2} \partial_{\vec{v}} \cdot \int d^3 v' \mathcal{Q}(\vec{v}, \vec{v}\,') \cdot (\partial_{\vec{v}} - \partial_{\vec{v}\,'}) f(\vec{v}) f(\vec{v}\,'), \tag{156}$$

where

$$\mathcal{Q} = 8\pi^4 \int d^3 k \frac{\vec{k}\vec{k}\, \varphi^2(k)}{|\epsilon(\vec{k}, \vec{k} \cdot \vec{v})|^2} \delta[\vec{k} \cdot (\vec{v} - \vec{v}')], \tag{157}$$

and $\varphi(k)$ is the Fourier transform of the potential. When comparing with the Landau-Fokker-Planck collision term we clearly see that they agree for $\epsilon \approx 1 + 1/k^2 \lambda_{De}^2$. Here

$$\epsilon = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \vec{k} \cdot \int \frac{\partial f_j / \partial \vec{v}}{\vec{k} \cdot \vec{v} - \omega} d^3 v \tag{158}$$

is the dispersion function, and the limit mentioned above corresponds to the static limit. The latter is equivalent to the Debye shielding. The Balescu-Lenard equation is more precise than the static limit: it takes care of the dynamical shielding of particles.

IV.5 Collision frequencies

On the basis of a kinetic equation we can study various phenomena in plasmas: transport, waves and instabilities, collisional and collisionless (Landau) damping, turbulence, and so on. Many of these aspects are investigated in this volume. In the following sections we shall consider some general estimates and tools without aiming for completeness. In general, it is very important to know the relevant collision frequencies. They have been mentioned several times already, and it is now time to figure them out from the collision term. Let us first make a few general remarks. In the kinetic equation for the one-particle distribution function f^α the collision term

$$K^\alpha = \sum_\beta K^{\alpha\beta} \tag{159}$$

consists of additive contributions of collisions with the various species. Particle numbers, total momentum, and total energy are conserved. We can define a friction force and a collisional rate of heat exchange

$$\vec{R}^\alpha = m_\alpha \int d^3 v\, \vec{v}\, K^\alpha \; , \tag{160}$$

$$Q^\alpha = \frac{1}{2} m_\alpha \int d^3 v\, |\vec{v} - \vec{u}^\alpha|^2 K^\alpha \; , \tag{161}$$

respectively, which satisfy the following relations

$$\vec{R}^e = \vec{R}^{ei} = -\vec{R}^{ie} = -\vec{R}^i \; , \tag{162}$$

$$Q^e = Q^{ei} = -Q^{ie} - \left(\vec{u}^e - \vec{u}^i\right) \cdot \vec{R}^{ei} \; , \tag{163}$$

$$Q^{ie} \approx -3n_i \frac{Z}{\tau_e} \frac{m_e}{m_i} (T_i - T_e) \; . \tag{164}$$

These relations express the conservation laws, but, in addition, show the weak energy exchange between electrons and ions. The latter fact is the reason for introducing electron and ion temperatures separately as plasma-dynamical variables.

The collision frequencies can be found in the simplest way from the linearized collision term. Like-particle collisions conserve the number, momentum, and energy of each species; they redistribute very efficiently momentum and energy among each other and are responsible for local (near) equilibrium states (after a time τ_α). Unlike-particle collisions exchange momentum and energy between species; they transfer energy between ions and electrons extremely slowly [characteristic time $(m_i/m_e)\tau_\alpha$]. Since for the relaxation times ($\tau \sim 1/\nu$)

$$\tau_{ee} \sim \tau_{ei} \sim \tau_e \quad (Z = 1) \; , \quad \tau_{ie} \gg \tau_{ii} \sim \tau_i \; , \tag{165}$$

holds, we are left with two characteristic times

$$\tau_e = \frac{3}{4\sqrt{2\pi}} \frac{m_e^{1/2} T_e^{3/2}}{n_i Z^2 e^4 \ln \Lambda} \; , \quad \tau_i = \frac{3}{4\sqrt{2\pi}} \frac{m_i^{1/2} T_i^{3/2}}{n_i Z^4 e^4 \ln \Lambda} \; . \tag{166}$$

Here, $\ln\Lambda \approx 13$ is the Coulomb logarithm. On these results one gets the collision frequencies which have can be found in standard textbooks. From here we conclude that a fusion plasma is not necessarily collisional. But because of confinement and trapping, although the mean free paths may be much longer than the machine dimensions, particles can stay long enough in the plasma before reaching the walls. Thus it can be expected that even weak collisions contribute to transport.

The derivation of the relaxation times is one of the main outcomes of kinetic theory. Looking at the parameter dependences, we recognize that the relaxation times increase with temperature. Then, from a very simple model we can obtain the following estimate for the resistivity:

$$\eta = \frac{m_e}{e^2 n \tau_e} \equiv \frac{1}{\sigma} \, . \tag{167}$$

Thus the resistivity of a plasma will decrease with temperature, which has severe consequences for nuclear fusion. The question how to obtain more exact results for the plasma (transport) coefficients belongs to transport theory.

IV.6 Linearized kinetics

One question now is how to solve the kinetic equations to get the relevant physical information. We cannot answer this question in general since we are dealing with nonlinear partial differential equations. In this section we concentrate on collective effects within linear response theory.

Let us investigate the initial value problem for the linearized kinetic equations, e.g. the Vlasov equation. Linearizing the Vlasov equation by assuming

$$f(\vec{q}, \vec{v}; t) = f_0(\vec{v}) + f_1(\vec{q}, \vec{v}; t), \tag{168}$$

where f_0 is the equilibrium distribution function which may depend on constants of motion, we obtain for the perturbation

$$\partial_t f_1 + \vec{v}\cdot\nabla f_1 - \frac{e}{m_e}\vec{E}_1 \cdot \partial_{\vec{v}} f_0 = 0, \tag{169}$$

$$\nabla\cdot\vec{E}_1 = -4\pi e \int d^3 v f_1. \tag{170}$$

For the reason of simplicity, this is written for the electrons when we can consider the ions as a smeared-out background, i.e. for high-frequency phenomena. Now a principal point has to be made. If we would solve the linear equations (169) and (170) by Fourier transformation in space and time, we would get the wrong (i.e. a misleading) answer. The reason is that the Fourier transform contains a lot of false modes which do not appear in a correct solution of the initial value problem. The latter is exactly what we have: We want to find out how an initial perturbation evolves, that is to say, whether it is amplified, damped, or disperses away. After Laplace transformation in time (still Fourier modes $\vec{k}$ are used in space, where for simplicity $\vec{k} = k\hat{x}$ is assumed),

$$F(p, \vec{v}) = \int_0^\infty f_1(\vec{v}; t) e^{-pt} dt, \tag{171}$$

$$f_1(\vec{v}; t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(p, \vec{v}) e^{pt} dp, \tag{172}$$

we obtain from (169) and (170)

$$(p + ikv_x)F(p, \vec{v}) - \frac{e}{m_e}E_{1x}(p)\frac{\partial f_0}{\partial v_x} = f_1(\vec{v}; 0), \tag{173}$$

$$ikE_{1x}(p) = -4\pi e \int F(p, \vec{v}) d^3 v, \tag{174}$$

where we have introduced the abbreviation $G(u) = \int dv_y \, dv_z \, f_1(u = v_x, v_y, v_z; 0)$ and a similar symbol g originating from f_0. This completes the formulation of the initial value problem for the linearized Vlasov equation. To get more insight, let us proceed as follows. Combining the last two equations, we obtain

$$E_{1x}(p) = -\frac{4\pi e}{ik} \frac{\displaystyle\int_{-\infty}^{+\infty} \frac{G(u)}{p + iku} du}{1 - \displaystyle\frac{\omega_{pe}^2}{k}\int \frac{dg}{du}\frac{du}{ku - ip}} \, , \tag{175}$$

which after back-transformation looks like

$$E_{1x}(t) = \frac{1}{2\pi i}\int_{\tilde{\sigma} - i\infty}^{\tilde{\sigma} + i\infty} E_{1x}(p) \, e^{pt} \, dp. \tag{176}$$

In evaluating this expression we have to know the contour for integration which is described by the Laplace transform in the initial value problem. In a nutshell, $\tilde{\sigma}$ has to be larger than $\Re p_\nu$ where p_ν designates the singularities of $E_{1x}(p)$ as given by (175). As is shown in Fig. 8, $E_{1x}(p)$ is only defined for $\Re p \geq \tilde{\sigma}$ In order to find the solution (176) by methods of function analysis it is necessary to know $E_{1x}(p)$ for all p. Then we have to use the analytic continuation of its denominator for all p. This, e.g., for $\Re p < 0$, is reached by choosing the u-contour as depicted in Fig. 8 (Landau contour). For $t \to \infty$ we then obtain

$$E_{1x}(t) \approx \mathrm{Res}(p)\exp(pt). \tag{177}$$

Note that the p used here is the pole with the largest real part. For its evaluation we have to discuss the dispersion relation

$$\epsilon(k, \omega = ip) := 1 - \frac{\omega_{pe}^2}{k^2}\int \frac{dg}{du}\frac{du}{u - \frac{\omega}{k}} = 0. \tag{178}$$

This has to be evaluated along the so called Landau contour. In the complex u-plane, the path has to be defined in such a way that all poles ip/k lie above the u-contour. For a Maxwellian equilibrium distribution and low damping ($\gamma/\omega_r \ll 1$), we can obtain from (178) the famous electron Landau damping rate

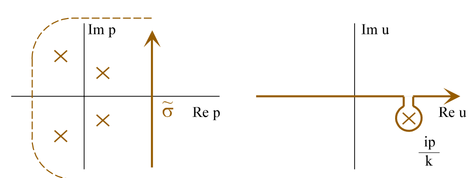

Figure 8: *Illustration of a Landau contour.*

$$\gamma \approx -\omega_{pe}\left(\frac{\pi}{8}\right)^{1/2}(k\lambda_{De})^{-3}\exp\left[-\frac{1}{2}(k\lambda_{De})^{-2} - \frac{3}{2}\right]. \tag{179}$$

Landau damping occurs due to phase mixing of free streaming solutions. The present analysis shows us that for small damping, i.e. $k\lambda_{De} \gg 1$, the plasma can react with some long living high-frequency wave-like phenomena. For a Maxwellian distribution we obtained an imaginary part of the frequency which corresponds to a (small) damping of the waves.

There are, however, many cases known where the distribution function is not Maxwellian, and instead of damping we can have (Landau) growth. A whole zoo of instabilities exists, and because of the many instabilities the dynamics of a plasma is rich and complex. In an unstable situation, the amplitudes of the excited modes grow, and very soon the validity of the linearization breaks down. The plasma becomes turbulent.

Let us now mention another point. Usually, for a Maxwellian the integral in (178) is traced back to the so called Z-function (or G-function)

$$G(\zeta) \equiv Z(-\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-p^2}}{\zeta - p}\, dp, \tag{180}$$

and the asymptotic behaviors of Z are well-known. Furthermore, the inclusion of the ion dynamics into the above derivation is straightforward. Therefore, for an electron-ion plasma the dispersion relation looks like

$$k^2 = \frac{\omega_{pe}^2}{2v_{te}^2} Z'(\zeta_e) + \frac{\omega_{pi}^2}{2v_{ti}^2} Z'(\zeta_i)\ . \tag{181}$$

For $\omega \geq \omega_{pe}$ (high-frequency or optical branch) and $\omega/k \gg v_{te}$ the solution is

$$\omega^2 \approx \omega_{pe}^2 \left(1 + 3k^2 \lambda_{De}^2\right), \tag{182}$$

i.e. electron plasma waves. Besides this high-frequency branch also a low-frequency (acoustic) mode exists which we obtain from (181) in the limit $\omega/k \ll v_{te}$ (leading to $Z'(\zeta_e) \approx -2$). The approximate solution for $\omega/k \gg v_{ti}$ is

$$\omega^2 \approx \frac{k^2 c_s^2}{1 + k^2 \lambda_{De}^2}\ , \tag{183}$$

where

$$c_s = \sqrt{(T_e + 3T_i)/m_i} \tag{184}$$

describes the so called ion-acoustic waves.

These examples show that in a plasma long-range (collective) phenomena (waves) can occur provided the corresponding damping is not too large. The particles "communicate" with each other via a mean field. For applications, we should have in mind that these results are derived from the linearized kinetic equations. They are valid as long as nonlinear (amplitude-dependent) processes are not important.

V. KINETIC DESCRIPTION OF STRONGLY MAGNETIZED PLASMAS

The idea of the following considerations is to simplify the kinetic description for a magnetized plasma (with $\rho_i/L \ll 1$). Considering the particles, we can subdivide the motion into a fast gyration (with small gyro-radii ρ) and a drift in the direction perpendicular to the magnetic field. The motion parallel to the magnetic field is more or less not hindered at all, and therefore we keep $v_\parallel$ as a independent variable. From a simplified point of view the exact particle positions are replaced by the gyro-centers, the perpendicular velocities are identified as the drift velocities, and $\vec{r}, v_\parallel, t$ remain as independent variables. [We should have in mind that the exact transformation to the so called drift-kinetic description has a few more technical aspects.] As we shall see in the next subsection, such a strategy leads to the drift-kinetic equation.

However, we should have in mind that by replacing the particle positions by their gyro-centers we cannot anymore resolve space scales of the order of the gyro-radius. Remember that we have a slow scale L (of the order of the toroidal minor radius or density gradient scale-length). Fast variations may occur for small, but not necessarily linear departures from the stationary state. If the wavelengths of the fast variations are of the order of the ion Larmor radius ρ_i, electric fields will also vary on the fast scale, and the particles see different field strengths during their gyro-motion. To take into account that effect is the purpose of the so called gyro-kinetic description being discussed in the second subsection. However, with respect to the latter we should mention that the full nonlinear theory is not yet available. The main reason is that when we subdivide f into

$$f = f_{slow} + \Delta f_{fast}\ , \tag{185}$$

$\Delta \ll 1$ is required for magnetized plasmas since otherwise for $\Delta \sim \mathcal{O}(1)$ and $\lambda_{fast} \sim \rho_i$ the plasma would be effectively demagnetized.

V.1 The drift-kinetic equation

As has been just mentioned, a kinetic equation for magnetized plasmas is much simpler than, e.g., the Landau-Fokker-Planck equation since it suppresses details on the gyro-radius scale. The guiding centers have a velocity

$$\vec{v}_{gc} = \hat{b} v_\parallel + \vec{v}_d\ , \tag{186}$$

where the drift velocity to lowest orders is

$$\vec{v}_d = \vec{v}_E + \frac{1}{\Omega}\, \hat{b}\ \times \left(\frac{\mu}{m}\ \nabla\ B + v_\parallel^2\, \vec{\kappa}\right)\ . \tag{187}$$

The latter consists of the $E \times B$-drift $\vec{v}_E$ and the curvature drift ($\vec{\kappa} = \hat{b} \cdot \nabla \hat{b}$). Note that according to the scaling

$$\Delta \equiv 0 \quad , \quad \frac{v_E}{v_{th}} \sim \delta \quad , \quad \partial_t \sim \delta\, \Omega \tag{188}$$

the polarization drift is of higher order. In (187) μ is the magnetic moment, and for its magnitude we have

$$\mu = \frac{m v_\perp^2}{2B} \qquad , \qquad \frac{d\mu}{dt} = \mathcal{O}(\delta)\ . \tag{189}$$

In other words, μ is an adiabatic invariant.

The total guiding-center energy is the sum of kinetic and potential energy,

$$U = \frac{m v_\parallel^2}{2} + \mu\, B + e\, \phi\ . \tag{190}$$

Note, that the last two terms represent the potential energy of a guiding center. Calculating the time change of the energy, we make the ansatz

$$\frac{d}{dt}\left[\frac{m}{2}\ v_\parallel^2\right] \approx \vec{v}_{gc} \cdot \left[e\vec{E} - \nabla(\mu\, B)\right] \tag{191}$$

for the time change of the kinetic energy of a guiding center.

The second term on the right-hand-side takes care of the mirror force $\vec{F}_m = -\nabla(\mu\, B)$.

One finds

$$\begin{aligned} \frac{dU}{dt} &\equiv \left(\frac{\partial}{\partial t} + \vec{v}_{gc} \cdot \nabla\right) U \\ &\approx e\,\frac{d\phi}{dt} + \mu\,\frac{dB}{dt} + \vec{v}_{gc} \cdot \left[e\vec{E} - \nabla(\mu\, B)\right] \\ &= e\,\frac{d\phi}{dt} + \mu\,\frac{dB}{dt} - \frac{e}{c}\,\vec{v}_{gc} \cdot \frac{\partial \vec{A}}{\partial t}\ . \end{aligned} \tag{192}$$

Without doing the exact calculation (see remarks below) we expect that the gyro-phase-averaged distribution function $\bar{f} = \bar{f}(\vec{r}, U, t)$ obeys the drift-kinetic equation

$$\frac{\partial \bar{f}}{\partial t} + \vec{v}_{gc} \cdot \nabla \ \bar{f} + \frac{dU}{dt} \ \frac{\partial \bar{f}}{\partial U} = 0 \ . \tag{193}$$

In that form of the drift-kinetic equation we have to insert the expression for dU/dt shown in the previous formula.

As was said already, a more satisfactory derivation of the drift-kinetic equation from the Landau-Fokker-Planck equation requires a more systematic procedure. Besides the transformation to new variables [from $\vec{r}, \vec{v}$ to $\vec{Y}$ (gyro-center), U, μ, and φ (gyro-phase)] the collision term has to be reconsidered. The basic idea is to rewrite the Landau-Fokker-Planck equation in the new variables and to recognize that the variation of the distribution function with respect to the gyro-phase φ is fast. That behavior suggests to make use of a multiple-time analysis. In addition, each part of the distribution function is being splitted into a gyro-phase averaged part and a rapidly oscillating part. When doing that (for a systematic derivation see R. Balescu, Transport Processes in Plasmas, Vol. 2, § 10) we arrive at

$$\begin{aligned} \frac{\partial \bar{f}}{\partial t} \quad &+ \quad \left(v_\parallel \hat{b} + \vec{v}_d\right) \cdot \nabla_Y \ \bar{f} - \frac{e}{c} \ v_\parallel \ \hat{b} \cdot \frac{\partial \vec{A}}{\partial t} \ \frac{\partial \bar{f}}{\partial U} \\ &\approx \quad \overline{K_0\{f_0, f_1\}} - \overline{K_1\{f_0, f_0\}} \ , \end{aligned} \tag{194}$$

where the right-hand-side represents the averaged linear collision term. Its evaluation is by no means trivial. Note that the space derivative is now with respect to the guiding-center position $\vec{Y}$. In addition, we have written the drift-kinetic equation for a species of electric charge e (including a sign). In general, we obtain drift kinetic equations for each species separately.

V.2 The gyro-kinetic approach

So far we have assumed that the fields are not varying rapidly (on the gyro-radius scale). However, it is possible that instabilities develop with $k_\perp \rho_i \sim \mathcal{O}(1)$. Let us first, for the reason of demonstration, assume that the perturbations are electrostatic, with the electric field $\vec{E}_1$. During its gyration, the particle will experience varying electric field strengths, resulting in an additional averaged $E \times B$ velocity

$$\langle \vec{v}_1 \rangle_{es} \approx \frac{c}{B_0} \left\langle \vec{E}_1(\vec{r}) \right\rangle \times \hat{b} \ . \tag{195}$$

Let us name the actual position of the particle as $\vec{r} = \vec{Y} + \vec{\rho}$, with

$$\vec{\rho} = -\frac{v_\perp}{\Omega} \ \vec{n}_2 = -\frac{v_\perp}{\Omega} \ (\cos \ \varphi \ \hat{e}_1 - \sin \ \varphi \ \hat{e}_2) \ . \tag{196}$$

Let us further consider one (linear) mode

$$\vec{E}_1(\vec{r}) = \vec{E}_1 \ e^{i\vec{k}_\perp \cdot \vec{\rho}} \equiv \vec{E}_1(\vec{Y}) \ e^{i\vec{k}_\perp \cdot (\vec{Y} + \vec{\rho})} \ . \tag{197}$$

Note, that the arguments specify the meaning of the amplitudes. We evaluate

$$\begin{aligned} \left\langle \vec{E}_1(\vec{r}) \right\rangle \quad &\equiv \quad \vec{E}_1 \int \frac{d\varphi}{2\pi} \ e^{i \frac{v_\perp}{\Omega} (k_2 \sin \ \varphi - k_1 \cos \ \varphi)} \\ &= \quad \vec{E}_1 \ J_0 \left(\frac{v_\perp}{\Omega} \ k_\perp \right) \equiv -i\vec{k}_\perp \ \phi_A \ J_0 \left(\rho k_\perp \right) \end{aligned} \tag{198}$$

when $k_1 = k_\perp \ \cos \psi$ and $k_2 = k_\perp \ \sin \psi$. Thus, for electrostatic perturbations we obtain the linearized gyro-kinetic equation

$$\begin{aligned} \frac{\partial \bar{f}_1}{\partial t} \quad &+ \quad \left(v_\parallel \hat{b} + \vec{v}_d\right) \cdot \nabla_Y \ \bar{f}_1 - \frac{e}{c} \ v_\parallel \ \hat{b} \cdot \frac{\partial \vec{A}}{\partial t} \ \frac{\partial \bar{f}_1}{\partial U} \\ &\approx \quad i \ \frac{c}{B_0} \ J_0 \left(\rho k_\perp \right) \phi_A \ (\vec{k} \times \hat{b}) \cdot \nabla_Y \ \bar{f}_0 \end{aligned} \tag{199}$$

in the collisionless approximation. Note that $\bar{f} = \bar{f}_0 + \bar{f}_1$, and $\bar{f}_1$ is the (small) response to the (electrostatic) rapid field perturbation $\vec{E}_1(\vec{r})$.

In the presence of (additional) rapid magnetic perturbations $\vec{B}_1(\vec{r})$, more first order terms appear. The solution of the linearized equation of motion

$$m \ \frac{d\vec{v}}{dt} = \frac{e}{c} \ \left[\vec{v}_1 \times \vec{B}_0 + \vec{v} \times \vec{B}_1\right] \tag{200}$$

is straightforward if we remember the derivation of the $E \times B$ drift:

$$\begin{aligned} \vec{v}_1 \quad &\approx \quad -\frac{1}{B_0} \ \hat{b} \times \left(\vec{v} \times \vec{B}_1\right) \\ &= \quad -\vec{v} \ \frac{\hat{b} \cdot \vec{B}_1}{B_0} + \vec{B}_1 \ \frac{\hat{b} \cdot \vec{v}}{B_0} \\ &= \quad -\vec{v}_\perp \ \frac{B_{1\parallel}}{B_0} + \vec{B}_{1\perp} \ \frac{v_\parallel}{B_0} \ . \end{aligned} \tag{201}$$

Thus, the additional averaged drift is

$$\begin{aligned} \langle \vec{v}_1 \rangle_{em} \quad &= \quad -\frac{1}{B_0} \ \left\langle B_{1\parallel} \ \vec{v}_\perp \right\rangle + \frac{v_\parallel}{B_0} \ \left\langle \vec{B}_{1\perp} \right\rangle \\ &\approx \quad -i \ \frac{v_\perp}{B_0} \ B_{A\parallel} \ J_1(\rho k_\perp) \ \hat{k}_\perp \times \hat{b} \\ &\qquad + i \ \frac{v_\parallel}{B_0} \ \vec{k}_\perp \times \hat{b} \ A_{A\parallel} \ J_0(\rho k_\perp) \ . \end{aligned} \tag{202}$$

Since the field will be also time-dependent $\sim e^{-i\omega t}$, the energy is no constant anymore, and we have to calculate

$$\begin{aligned} &\left\langle \frac{dU}{dt} \right\rangle = -i \ \omega \ e \left[\langle \phi_1 \rangle - \frac{1}{c} \left\langle \vec{v} \cdot \vec{A}_1 \right\rangle \right] \\ &\approx -i \ \omega \ e \left[J_0(k_\perp \rho) \phi_A - i \ \frac{v_\perp}{c} \ J_1(k_\perp \rho) (\hat{k}_\perp \times \hat{b}) \cdot \vec{A}_{A\perp} \right. \\ &\qquad \left. - \ \frac{v_\parallel}{c} \ J_0(k_\perp \rho) A_{A\parallel} \right] \ . \end{aligned} \tag{203}$$

The following additional terms

$$- \langle \vec{v}_1 \rangle_{em} \cdot \nabla_Y \ \bar{f}_0 - \langle \vec{v}_1 \rangle_{es} \cdot \nabla_Y \ \bar{f}_0 - \left\langle \frac{dU}{dt} \right\rangle \ \frac{\partial \bar{f}_0}{\partial U} \tag{204}$$

appear on the right-hand-side of the linearized gyro-kinetic equation. They can be combined, finally leading to

$$\begin{aligned} &\frac{\partial \bar{f}_1}{\partial t} + \left(v_\parallel \ \hat{b} + \vec{v}_d\right) \cdot \nabla_Y \ \bar{f}_1 - \frac{e}{c} \ v_\parallel \ \hat{b} \cdot \frac{\partial \vec{A}}{\partial t} \ \frac{\partial \bar{f}_1}{\partial U} \\ \approx \quad &i \left[J_0(k_\perp \rho) \left\{ \phi_A - \frac{v_\parallel}{c} \ A_{A\parallel} \right\} + \frac{v_\perp}{ck_\perp} \ J_1(k_\perp \rho) B_{A\parallel} \right] \\ &\times \left\{ e \ \omega \ \frac{\partial \bar{f}_0}{\partial U} + \frac{c}{B_0} \left(\vec{k}_\perp \times \hat{b}\right) \cdot \nabla_Y \ \bar{f}_0 \right\} \ . \end{aligned} \tag{205}$$

This is the (collisionless) linear gyro-kinetic equation for rapid perturbations (on the gyro-radius scale). As has been mentioned already, when large disturbances take place the linearization breaks down and we have to return to the (unreduced) description by the Landau-Fokker-Planck or Balescu-Lenard equations.

V.3 Simple transport truncations

Here, we shall not present systematic transport theory which is based on kinetic equations. But sometimes a simple ansatz for the solution of a kinetic equation is quite successfully used in moment equations. Similar to Sec. III.2 one can define hydrodynamic variables. By integration one can derive exact – although not closed – equations. Let us do it here for the plasma-dynamical variables $n_\alpha(\vec{x},t) = \int d^3v f^\alpha(\vec{v},\vec{x},t)$, $n_\alpha(\vec{x},t)\vec{u}_\alpha(\vec{x},t) = \int d^3v \vec{v} f^\alpha(\vec{v},\vec{x},t)$, $n_\alpha T_\alpha = \frac{1}{3}m_\alpha \int d^3v |\vec{v}-\vec{u}^\alpha|^2 f^\alpha(\vec{v},\vec{x},t)$. The scalar pressure is $p_\alpha = n_\alpha T_\alpha$.
Straightforward integration of the Landau-Fokker-Planck equation leads to

$$\partial_t n_\alpha + \nabla \cdot (n_\alpha \vec{u}^\alpha) = 0 \; , \qquad (206)$$

$$\partial_t \left(m_\alpha n_\alpha u_r^\alpha\right) + \nabla_m \left(m_\alpha n_\alpha u_r^\alpha u_m^\alpha + \delta_{rm} n_\alpha T_\alpha + \Pi_{rm}^\alpha\right)$$
$$-e_\alpha n_\alpha \left(E_r + \frac{1}{c}\epsilon_{rmn} u_m^\alpha B_n\right) = R_r^\alpha \; , \qquad (207)$$

$$\begin{aligned} n_\alpha \partial_t T_\alpha &= -n_\alpha \left(\vec{u}^\alpha \cdot \nabla\right) T_\alpha - \frac{2}{3} n_\alpha T_\alpha \nabla \cdot \vec{u}^\alpha \\ &\quad -\frac{2}{3}\Pi_{mn}^\alpha \nabla_m u_n^\alpha - \frac{2}{3}\nabla_m q_m^\alpha + \frac{2}{3} Q^\alpha \; , \end{aligned} \qquad (208)$$

where

$$R_r^\alpha \equiv m_\alpha \int K^\alpha v_r d^3v \quad , \quad Q^\alpha \equiv \frac{1}{2} m_\alpha \int K^\alpha |\vec{v}-\vec{u}^\alpha|^2 d^3v \; . \qquad (209)$$

Also, the dissipative part of the pressure tensor

$$\begin{aligned} \Pi_{rs}^\alpha(\vec{x},t) &= m_\alpha \int d^3v [v_r - u_r^\alpha][v_s - u_s^\alpha] f^\alpha(\vec{v},\vec{x},t) \\ &\quad -\frac{1}{3} m_\alpha \delta_{rs} \int d^3v |\vec{v}-\vec{u}^\alpha|^2 f^\alpha(\vec{v},\vec{x},t) \end{aligned} \qquad (210)$$

and the heat current

$$q_r^\alpha(\vec{x},t) = \frac{1}{2} m_\alpha \int d^3v [v_r - u_r^\alpha] |\vec{v}-\vec{u}^\alpha|^2 f^\alpha(\vec{v},\vec{x},t) \qquad (211)$$

are still undetermined. Dynamic equations for the mass density ρ, the charge density σ, the mean mass velocity $\vec{u}$, the electric current density $\vec{j}$ follow from here by taking sums or differences. Note that all these equations still contain the exact distribution function(s) $f^\alpha(\vec{v},\vec{x},t)$. For a further simplification, one considers only phenomena taking place on a hydrodynamic scale being much slower than the collisional time scale. Then the distribution function(s) $f^\alpha(\vec{v},\vec{x},t)$ can be assumed to be close to a Maxwellian.

Ignoring the systematic procedure of transport theory, we discuss two popular simplifications in some ad hoc manner. We can order the dominating $E \times B-$velocity $\vec{u}_E$ in two different ways: First, the MHD ansatz $u_E \sim v_{th}$ with the distribution function

$$f \approx f_M\left(\vec{v}-\vec{u}\right) + \mathcal{O}(\delta) \; , \qquad (212)$$

leading to the MHD equations

$$\begin{aligned} \partial_t \rho + \nabla(\rho \vec{u}) &= 0 \; , & (213) \\ \partial_t(\rho\vec{u}) + \nabla \cdot (\rho \vec{u}\vec{u}) + \nabla P &= \frac{1}{c}\vec{j} \times \vec{B} \; , & (214) \\ \partial_t P + \vec{u} \cdot \nabla P + \frac{5}{3} P \nabla \cdot \vec{u} &= 0 \; , & (215) \\ \vec{E} + \frac{1}{c}\vec{u} \times B &= \eta \vec{j} \; , & (216) \\ \nabla \times \vec{B} &= \frac{4\pi}{c}\vec{j} \; , & (217) \\ \nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \; . & (218) \end{aligned}$$

Second, the drift truncation: $u_E \sim \delta v_{th}$ with the distribution function

$$f \approx f_M(\vec{v}) \left[1 + 2\frac{u_\parallel v_\parallel}{v_{th}^2}\right] - \vec{\rho} \cdot \nabla f_M + \mathcal{O}(\delta^2) \; . \qquad (219)$$

The assumed distribution function deviates from a Maxwellian by two terms. The first one yields the parallel velocity while the second one leads to the drift velocities

$$\begin{aligned} \int d^3v \left(-\vec{\rho} \cdot \nabla f_M\right) \vec{v}_\perp &= \frac{P}{m\Omega}\vec{b} \times \left(\nabla \ln P + \frac{e\nabla\phi}{T}\right) \\ &\equiv n\,\vec{u}_\perp \; . \end{aligned} \qquad (220)$$

In summary, using the drift ordering in all equations, we finally arrive at the following (drift) model

$$\frac{dn}{dt} + n \nabla \cdot \vec{u} = 0 \; , \qquad (221)$$

$$\frac{dP}{dt} + \frac{5}{3} P \nabla \cdot (\vec{u} - \vec{v}_{pi}) = 0 \; , \qquad (222)$$

$$m_i n \left[\frac{d\vec{u}_E}{dt} + \frac{d}{dt} \;\; \vec{b} u_\parallel - \vec{v}_{pi} \cdot \nabla \vec{b} u_\parallel\right]$$
$$+\nabla P - \frac{1}{c}\vec{j} \times \vec{B} = 0 \; , \qquad (223)$$

$$\vec{E} + \frac{1}{c}\vec{u} \times \vec{B} - \frac{1}{en}\left(\frac{1}{2}\nabla P - \frac{1}{c}\vec{j} \times \vec{B}\right)$$
$$-\eta\left[\vec{j} - \frac{3}{4}\frac{cn}{B}\vec{b} \times \nabla\frac{P}{n}\right] = 0 \; . \qquad (224)$$

The relations

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} \quad , \quad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \; , \qquad (225)$$

$$\frac{d}{dt} = \partial_t + \vec{u} \cdot \nabla \quad , \quad \vec{u}_E = c\,\frac{\vec{E} \times \vec{B}}{B^2} \qquad (226)$$

have to be added. The ion polarization drift is defined as

$$\vec{v}_{pi} = \frac{1}{2\Omega_i m_i n}\vec{b} \times \nabla P \; . \qquad (227)$$

In contrast to the initial value problem of the linearized kinetic equations, the last two cases did not provide us with an explicit solution of the kinetic equation(s). They "only" simplified the problem by changing from the detailed description in configuration and velocity space (via the distribution functions f^α) to a reduced one only in configuration space (via a finite number of moments n_α, $\vec{u}^\alpha$, T_α, ...). When we are not interested in detailed wave-particle resonances, and when slow (hydrodynamic) processes are of interest, such a reduction is advisable. The momentum equations still contain important informations, e.g. about nonlinear processes, collective effects, etc.

VI. CONCLUDING REMARKS

This overview covers only a small part of the huge number of concepts and results which can be found in basic statistical plasma physics. The reader is advised to consult the basic literature [1–24] for a deeper understanding of the fundamental aspects.

Non-equilibrium statistical plasma physics is an unfinished agenda. It is strongly recommended to read the current literature when more specific questions, e.g. on the gyro-kinetic approach or hybrid models (just to name a few), are being posed.

Nevertheless, the available kinetic plasma approaches have been quite successful in the past. Hot plasmas occurring in magnetic fusion devices can be often described by classical and non-relativistic physical statistics of non-equilibrium systems. The resulting kinetic equations, being closed, but coupled equations for the one-particle distribution functions for the various species, contain individual as well as collective effects. The latter are caused by the simultaneous action of many particles via the long-range Coulomb forces and generate quasi-macroscopic fields. The individual effects are caused by small-distance (more or less) binary collisions. The nature of their fields is essentially chaotic, and they are responsible for irreversible processes.

The kinetic descriptions are mainly used for three purposes: First, they are appropriate to analyze equilibrium and stationary states, respectively. Secondly, they allow to determine the dynamical behaviors (at least near stationary states) in the forms of waves and instabilities. And finally, they are the basis for transport theory. Because of the rich and generally fast dynamics of a fusion plasma, transport is one of the key questions of thermonuclear fusion. Nonconvective transport is either caused by (chaotic) individual collisions or by (nonlinear) fluctuating collective fields. The discussion of that fascinating area is beyond the scope of this contribution.

Other concepts than magnetic fusion, e.g. laser-driven fusion, require a more thorough analysis of dense (strongly coupled or non-ideal) plasmas. In laser-material interaction problems, also ultra-fast non-equilibrium phenomena are of fundamental interest. Again, that is another topic.

ACKNOWLEDGMENTS

This work was performed under the auspices of the Graduiertenkolleg “Hochtemperatur-Plasmaphysik” and the Sonderforschungsbereich “Universelles Verhalten gleichgewichtsferner Plasmen: Heizung, Transport und Strukturbildung”. Continuous discussions with the colleagues at the Forschungszentrum Jülich are gratefully acknowledged.

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