KINETIC THEORY OF PLASMA WAVES - Part I: Introduction

P.U. Lamalle
Laboratoire de Physique des Plasmas - Laboratorium voor Plasmafysica
EURATOM Association, Partner in the Trilateral Euregio Cluster
École Royale Militaire - Koninklijke Militaire School, B-1000 Brussels, Belgium

ABSTRACT

The kinetic description of linear waves in plasmas is succinctly presented, with emphasis on applications to high-frequency (hf) wave heating and current drive. The Maxwell-Vlasov system of equations is introduced. Its two-timescale analysis yields the linearized Vlasov and the quasilinear Fokker-Planck equations. The standard guiding centre and Hamiltonian formalisms are presented. Two formulations of the hf plasma wave equation are given: as a partial differential equation to hold at each position, and as a global Galerkin ('variational') form.

I. INTRODUCTION

The various plasma wave heating and current drive methods used in magnetic confinement machines [1, 2] are based on resonant collisionless absorption of electromagnetic energy by certain classes of particles. Several types of wave-particle interactions exist, with resonance conditions of the form

$$\omega - \mathbf{k}.\mathbf{V} \equiv \omega - k_{//} v_{//} - \mathbf{k}_{\perp}.\mathbf{v}_{dr} = p\omega_{cs} \qquad (1)$$

In this expression[1], ω is the angular frequency of the quasi-monochromatic waves launched by an external system of antennae (time dependence $\propto e^{-i\omega t}, \omega > 0$), and $\mathbf{k} = k_{//}\mathbf{e}_{//} + \mathbf{k}_{\perp}$ is the wavevector of one wave mode, with a component $k_{//}$ along the equilibrium magnetic induction $\mathbf{B_0} = B_0\mathbf{e}_{//}$. The particle's guiding centre (gc) velocity,

$$\mathbf{V} = v_{//}\mathbf{e}_{//} + \mathbf{v}_{dr} \qquad (2)$$

has its main component $v_{//}$ along $\mathbf{B_0}$, the drift velocity $\mathbf{v}_{dr}$ (orthogonal to $\mathbf{B_0}$) being usually much smaller. The left-hand side of equation (1) is thus the Doppler-shifted wave frequency of the mode under consideration as seen in a frame of reference moving with the gc. Each particle species, ion or electron, is labeled with the index s, has a charge q_s, a rest mass m_{s0} and a relativistic mass m_s:

$$m_s = \gamma m_{s0}, \qquad \gamma = 1/\sqrt{1-(\mathrm{v}/c)^2} \qquad (3)$$

[1] To be critically be re-examined for inhomogeneous plasmas, cf. equation (31') and Part III.

Relativistic effects are required for an accurate description of electron heating and current drive, as shown in the next lecture [3], and they will be taken into account in the following; a classical treatment is however fully adequate for the much heavier ions (note v/c~0.04 for a 3.5MeV fusion alpha particle). The relativistic cyclotron frequency of the particle, i.e. the angular frequency of its rapid gyration perpendicular to $\mathbf{B_0}$ around the gc, is given by

$$\omega_{cs} = q_s B_0 / m_s \qquad (4)$$

(With this definition, the cyclotron frequency is positive for the ion species and negative for the electrons.) Implicit in writing equation (1) is the drift approximation,

$$\varepsilon_D \equiv \rho_L / L << 1 \qquad (5)$$

whereby the Larmor radius $\rho_L = v_{\perp} / \omega_{cs}$ is assumed much smaller than any characteristic length of variation of the magnetic equilibrium, L [4].

Last but not least, the integer p in the resonance condition is the cycloton harmonic index, and labels the type of interaction: p = 0 corresponds to the Landau-Cerenkov interaction, giving rise among other effects to Landau damping [5]. In this case, resonance takes place when the Doppler shift **k.V** equals the wave frequency. Nonzero values of p correspond to cyclotron interactions of various orders, associated with cyclotron damping: interaction at the fundamental cyclotron frequency ω_{cs} for p=±1, and at its harmonics for |p|>1. In usual cyclotron resonance configurations, the Doppler shift is much smaller than the wave and cyclotron frequencies, and resonance is only possible for $p\, q_s > 0$ ('normal' cyclotron damping); nevertheless, for high-energy particles, the Doppler shift may be so large that 'anomalous' cyclotron damping occurs in the case $p\, q_s < 0$.

In linear wave theory, each type of resonant interaction involves specific components of the particle velocity; broadly speaking, resonance occurs when associated components of the high frequency fields are constant in a frame of reference attached to the particle. Equation (1) expresses this statement in guiding centre variables (note that the resonances at the harmonics, |p|>1, require significant spatial variations of the hf fields on the Larmor radius scale).

Linear kinetic instabilities in a plasma are characterized by the same resonance conditions as plasma wave heating, but, conversely to the latter, result in a net transfer of energy from particles to waves and amplification of the latter. These opposite directions of energy transfer find their origin in differences in the plasma reference state (e.g. sign of the slope of the distribution function at resonance). The two areas of research may roughly be compared as follows: in plasma heating experiments, the wave frequency is imposed by the hf generators (it is a real number); the waves are damped spatially as they propagate inside the plasma; one analyzes their propagation and absorption by solving boundary value problems ('full-wave' description), or initial value problems (in the geometric optics approximation). In the study of microinstabilities, one is interested in the complex spectrum of eigenfrequencies of a plasma configuration, their imaginary part giving the growth rate of the corresponding eigenmode with time; theoretical analysis relies on the solution of eigenvalue problems.

Why use a kinetic theoretical description of waves in hot plasmas? A calculation based on a multi-fluid description, such as cold plasma wave theory, is much simpler; however, it cannot account in detail for the resonant wave-particle interactions of equation (1), that only take place for particular values of the velocity components within the broad range available in a hot particle population. Moreover, in all experiments, the plasma is inhomogeneous, and, exception made for the wave frequency, the coefficients of the resonance condition (1) vary with position. Under intense wave heating, such velocity-dependent interactions modify the distribution functions, producing e.g. high-energy particle tails and strong velocity anisotropy. The fate of the wave power transferred to the resonant particles ultimately depends on their collisional relaxation against the background plasma. This is once more a process with a strong velocity dependence. Still more, the finite Larmor radius of the particles gives birth to additional modes of wave propagation (the Bernstein modes, presented in Westerhof's lecture [3]), which have no counterpart in a cold plasma description. Hence there is abundant motivation for a kinetic wave theory, taking the individual motions of all particles into account.

II. THE MAXWELL-VLASOV SYSTEM [6, 7]

A. The electromagnetic field satisfies Maxwell's equations: in SI units,

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\times\mathbf{H} = \mu_0\left(\sum_s \mathbf{j}_s + \mathbf{j_a}\right) + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{6}$$

$$\boldsymbol{\nabla}.\mathbf{B} = 0, \qquad \boldsymbol{\nabla}.\,\mathbf{E} = \left(\sum_s \rho_s + \rho_a\right)/\varepsilon_0 \tag{7}$$

where (ρ_s, $\mathbf{j}_s$) are the charge and current densities due to particle species s, and (ρ_a, $\mathbf{j}_a$) are due to external sources (e.g. the currents flowing on an array of ICRH antennae). Charge conservation for the sources reads

$$\boldsymbol{\nabla}.\mathbf{j_a} + \frac{\partial\rho_\mathbf{a}}{\partial t} = 0 \tag{8}$$

A similar relation holds for each species, with eventual additional terms for sources and sinks of particles. The various particle species are described by their distribution functions:

$$f_s = f_s(\mathbf{r},\mathbf{p},t) \tag{9}$$

At every instant t, this function is the density of particles of type s at the observation point **r**, having a momentum **p**. Its evolution is described by the Boltzmann kinetic equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\nabla f_s + q_s(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\frac{\partial f_s}{\partial\mathbf{p}} = \sum_{s'} C(f_s, f_{s'}) + S \tag{10}$$

The left hand side has the form of a total time derivative along the orbit of a particle subject to the Lorentz force: its equations of motion are indeed

$$\dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{p}} = q_s(\mathbf{E}+\mathbf{v}\times\mathbf{B}), \quad \mathbf{p} = m_s\mathbf{v} \tag{11}$$

hence equation (10) can also be written

$$\frac{d}{dt} f_s = \sum_{s'} C(f_s, f_{s'}) + S \tag{12}$$

The Landau collision term C(,) is an integro-differential operator of Fokker-Planck type describing binary Coulomb interactions at close range[2], either between different particle species ($s' \neq s$) or between like particles ($s'=s$). Its explicit form can be found in [7] in the nonrelativistic case, relativistic expressions being given in [8]. Collisions yield a first source of nonlinearity in the Boltzmann equation, and also introduce irreversibility. Their effect on the long term, in the absence of a counteracting factor such as wave heating, is to drive the various distribution functions toward Maxwellians.

The last term on the right-hand side, S, accounts for eventual particle sources and losses, such as resulting from beam injection, nuclear reactions, ionization of neutrals.

A very distinctive property of the kinetic equation (10), when compared e.g. with the Boltzmann equation for a neutral gas, is the presence of the self-consistent electromagnetic field: the source terms in Maxwell's equations include the charge and current densities of each plama species, themselves moments of the distribution functions:

$$\rho_s(\mathbf{r},t) = q_s \int f_s(\mathbf{r},\mathbf{p},t)\, d^3p \tag{13}$$

$$\mathbf{j}_s(\mathbf{r},t) = q_s \int f_s(\mathbf{r},\mathbf{p},t)\, \mathbf{v}\, d^3p \tag{14}$$

The dependence of **E** and **B** on all the f_s renders the kinetic equations (10) strongly mutually coupled and highly nonlinear. This self-consistency was first introduced in the kinetic equation by Vlasov, whose name is attached to the

[2] I.e. within a Debye screening length.

collisionless and source-free version of the above Boltzmann equation:

$$\frac{d}{dt} f_s = 0 \qquad \text{(Vlasov equation)} \qquad (15)$$

The present lecture addresses plasmas in the long mean-free-path regime, where the times between successive collisions experienced by a particle are much longer than its characteristic times of transit across the region of space under study. The typical wave frequencies are also much larger than the collision frequencies. The Vlasov equation is thus fundamental to the theory of hf wave propagation, whereas the collision term of the Boltzmann equation 'enters the stage' on a much longer timescale, redistributing to the bulk plasma the energy absorbed from the waves by selected classes of particles. One *caveat* to the Vlasov equation will be discussed in the third part of these lectures: in the bounded magnetic configurations of fusion experiments, we shall see that even infrequent collisions play an important role, randomizing the phase relationships between waves and the various oscillations that characterize particle motion.

Equation (15) states the conservation of f_s along the particle trajectories (given by eq.(11)) for Vlasov plasmas. It can also be seen as an assertion of incompressibility for a 6-dimensional 'phase fluid' in (**r**, **p**) space. Note that, in arbitrary fields **E**, **B**, the particle orbits are generally not integrable: their motion has no first integral (such as e.g. the conservation of energy) and is completely stochastic; the orbits cover most of phase space ergodically. Moreover the electromagnetic fields are not known *a priori*, but depend on the dynamical history of all the particles! As formulated above, the problem is thus a formidable one: to solve a system of nonlinear coupled kinetic equations in 6 independent phase-space variables plus the time, coupled with the 3D Maxwell equations. Scales of variation therein may range over several orders of magnitude, as witnessed for instance by the drift ordering (5), or by the relatively slow timescale of evolution of a plasma equilibrium (e.g. ~50ms) subject to various auxiliary heating methods using waves with periods ranging from ~20ns (ion cyclotron), ~1ns (lower hybrid) down to ~10ps (electron cyclotron).

Drastic (but physically sound) simplifications and analytical progress are thus imperative. In the initial plasma state, prior to auxiliary wave heating, the distribution functions are Maxwellians, or nonmaxwellians of known shape (this is e.g. the case for injected beams, or fusion alpha particles). Some of the species may not interact with the waves for the heating scheme under study, and can be treated as a background against which the resonant species relax collisionally. Rapid oscillations superimposed on a slow phenomenon, or very short wavelengths in a weakly inhomogeneous medium, usually intractable by brute numerical force, are providential for theoreticians: they allow application of a panoply of asymptotic methods [9], which accuracy increases with the 'wildness' of the oscillation. The two-timescale analysis of the Boltzmann equation outlined in Section III, and the saddle point methods used in evaluating trajectory integrals such as in equation (30) below are two important examples. The approximation of geometric optics leads to powerful ray-tracing methods that allow 3-dimensional numerical investigation of the propagation and absorption of centimetric or millimetric waves in large machines (the major radius of the JET tokamak is ~3m).

Let us mention that massively parallel computers open a way to direct numerical solution of the nonlinear kinetic equation by following the motions of huge numbers of particles, interacting through a self-consistent field and collisions. In the drift kinetic regime, which corresponds to low frequencies ($\omega << \omega_{cs}$) and wavelengths larger than a typical Larmor radius, it is not necessary to describe the rapid cyclotron gyration in detail, and one solves a kinetic equation for the gyroangle-averaged distribution function. For instance, direct 3-D simulations of the transport induced by electrostatic turbulence (i.e. assuming **B**=0, $\mathbf{E} = -\boldsymbol{\nabla}\Phi$ for the fluctuating fields) have been reported [10], in which 10^8 guiding centre trajectories are followed over 2 10^4 time steps in toroidal geometry.

B. As we are interested in high-frequency waves and cyclotron resonances, we must take the fast gyration of particles fully into account. The analytical developments take advantage of the widely different timescales characterizing wave propagation on the one hand, evolution of the reference state under heating and collisions on the other. This allows linearization of the kinetic equation around a slowly evolving reference state, itself described by the quasilinear Fokker-Planck equation (Sections III, IV, respectively). Different sets of variables have been used to deal with the present problem. They have in common the extraction of the fast gyrophase ϕ. Figure 1 and the following equation remind the decomposition of particle velocity and position in gc variables [4]. $\mathbf{n}_1$, $\mathbf{n}_2$ are unit vectors orthogonal to $\mathbf{B_0}$, following the cyclotron gyration.

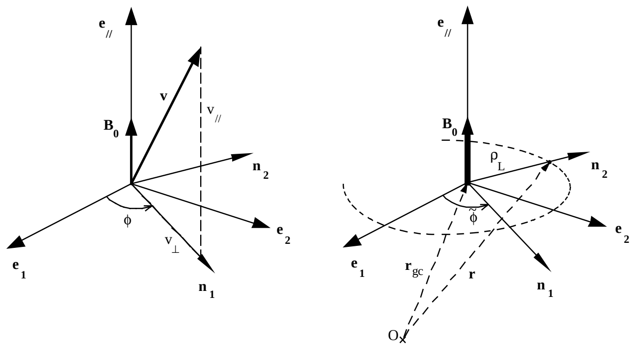

Fig. 1 - Left: components of the particle velocity. Right: relative positions of particle (**r**) *and guiding centre* ($\mathbf{r_{gc}}$).

$$\mathbf{r} = \mathbf{r_{gc}} + \rho_L \mathbf{n}_2 \,, \quad \mathbf{v} = \mathbf{V} + v_\perp \mathbf{n}_1 \,, \qquad \dot{\phi} = -\omega_{cs} + O(\varepsilon_D)$$

In Section III we outline the solution of the Vlasov equation for high frequency perturbations, introducing the two main methods used in the literature: the trajectory integral and the Hamiltonian formalisms (there are other approaches, but we cannot cover them here). The trajectory integral method [e.g. 6], which is very intuitive, has been applied to homogeneous and weakly inhomogeneous plasmas since the dawn of kinetic wave theory. More recently, it has also been applied to more complex geometries such as the inhomogeneous slab [e.g. 11] and the tokamak, with various degrees of approximation. The Hamiltonian formalism was applied to inhomogeneous axisymmetric plasmas by Kaufman in 1972 [12], providing very elegant but rather abstract expressions. Its strength resides in underlying powerful results from advanced dynamics. These general expressions were later simplified [13], and implemented in a numerical code. Let us stress that the two approaches are physically equivalent, as they only differ in form. Choosing one or the other is ultimately a matter of personal preference, but the trajectory integral formulation, based on standard guiding centre variables, is by all means more transparent. A detailed comparison and additional references are given in [14].

C. To proceed with the Hamiltonian formalism, we need state a few results. Introducing the electromagnetic field potentials $\mathbf{\Phi}$ and $\mathbf{A}$,

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A} \qquad (16),$$

the Lorentz force on a particle can be expressed in terms of the Lagrangian potential $\mathcal{M}$:

$$q_s(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = (\frac{d}{dt}\frac{\partial}{\partial \mathbf{v}} - \nabla)\,\mathcal{M} \qquad (17)$$

$$\mathcal{M} = q_s(\Phi - \mathbf{A}.\mathbf{v}) \qquad (18)$$

The Lagrangian of the particle is then [15]

$$\mathcal{L} = -m_{s0}c^2/\gamma - \mathcal{M} \ (19)$$

and its Hamiltonian

$$\mathcal{H} = m_s c^2 + q_s\Phi = \sqrt{m_{s0}^2 c^4 + c^2(\mathbf{P}-q_s\mathbf{A})^2} + q_s\Phi \qquad (20)$$

where $\mathbf{P} = \partial\mathcal{L}/\partial\mathbf{v} = \mathbf{p} + q_s\mathbf{A}$ is the generalized momentum. The dynamics is then given by

$$\dot{\mathbf{r}} = \partial\mathcal{H}/\partial\mathbf{P}, \qquad \dot{\mathbf{P}} = -\partial\mathcal{H}/\partial\mathbf{r}, \qquad \dot{\mathcal{H}} = \partial\mathcal{H}/\partial t \quad (21)$$

Integrable Hamiltonian systems, such as a particle in a uniform and constant magnetic field, possess by definition enough first integrals to characterize their trajectories algebraically (3 constants of the motion are required for a single particle). Two results from the theory of dynamical systems are fundamental in the present context [16, 17]:

- The existence of adiabatic invariants, i.e. quantities which remain constants of the motion when the system is 'slightly' perturbed.
- For conservative systems (i.e. for which $\mathcal{H}$ is a constant, the total energy), the existence of action-angle variables, such that the actions $\mathbf{J}$ are constants of the motion, and the canonically conjugate angles $\boldsymbol{\theta}$ are linear functions of the time: in terms of these variables, Hamilton's equations become

$$\mathcal{H} = \mathcal{H}(\mathbf{J})$$

$$\dot{\mathbf{J}} = -\partial\mathcal{H}/\partial\boldsymbol{\theta} \equiv 0, \qquad \dot{\boldsymbol{\theta}} = \partial\mathcal{H}/\partial\mathbf{J} \equiv \boldsymbol{\Omega}(\mathbf{J}) \qquad (22)$$

A 'near-integrable' system is a small perturbation of an integrable system, for instance a particle moving in weakly inhomogeneous, slowly varying fields in the drift approximation of equation (5). The above results also hold for 'most' orbits of a near-integrable system. The magnetic moment is the best known example of adiabatic invariant in the present context.

D. We now address the simplification of the kinetic equation (10). A two-timescale analysis is carried out: t_0 characterizes the 'slow' evolution of the plasma reference state under heating and collisions, and t_{-1} is the 'rapid' timescale associated with the wave and cyclotron periods. One extracts the rapid time dependence as follows:

$$f_s(\mathbf{r},\mathbf{p},t) = f_{s0}(\mathbf{r},\mathbf{p},t_0) + \mathrm{Re}\left[f_s(\mathbf{r},\mathbf{p}) e^{-i\omega t_{-1}} \right] \quad (23)$$

with a similar decomposition for the electromagnetic fields and potentials. This asymptotic analysis [9, 18] treats the two times as independent variables. In contrast with standard perturbation methods, where f_{s0} would be independent of the time and higher order terms retained, the two-timescale approach allows avoidance of secular terms in the expansions; one thus obtains an equation for the 'slow' evolution, the quasilinear Fokker-Planck equation, valid over a much longer time. Henceforth, we use the subscript '0' to indicate the reference state, we work with the complex amplitudes at ω for all hf quantities (with no subscript), and keep the uniform notation 't' for the time. For the hf potentials, one chooses the radiation gauge $\Phi = 0$, hence from equation (16),

$$\mathbf{E}(\mathbf{r},\omega) = i\omega\mathbf{A}(\mathbf{r},\omega) \qquad (24)$$

III. THE LINEARIZED VLASOV EQUATION

This section deals with the rapid timescale t_{-1}. Linearizing equation (15), one introduces the derivative along the unperturbed particle trajectory (i.e. its orbit in the absence of waves):

$$\frac{D}{Dt} \equiv \left.\frac{d}{dt}\right|_0 = -i\omega + \mathbf{v}.\nabla + q_s(\mathbf{E_0} + \mathbf{v}\times\mathbf{B_0}).\frac{\partial}{\partial\mathbf{p}} \qquad (25)$$

The linearized Vlasov equation is then

$$\frac{D}{Dt} f_s = -q_s(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\frac{\partial f_{s0}}{\partial \mathbf{p}}\Bigg| \qquad (26)$$

The right-hand side comes from the hf Lorentz force, which produces a first-order correction to the orbit equation.

In action-angle variables, taking $f_s = f_s(\boldsymbol{\theta}, \mathbf{J}, t)$, one simply obtains

$$(\boldsymbol{\Omega}.\frac{\partial}{\partial \boldsymbol{\theta}} - i\omega)\ f_s = -\delta\dot{\mathbf{J}}.\frac{\partial f_{s0}}{\partial \mathbf{J}} \qquad (27)$$

The perturbed actions in the right-hand side are non-trivial quantities, resulting from the linearized perturbation $\delta\mathcal{H}$ of the Hamiltonian by the waves. The latter is easily obtained from equations (20), (24):

$$\delta\mathcal{H} = q_s\ \mathbf{E}.\mathbf{v}/(-i\omega) = \mathfrak{N}(\mathbf{r},\mathbf{v},\omega) \qquad (28)$$

$$\delta\dot{\mathbf{J}} = -\partial\,\delta\mathcal{H}/\partial\boldsymbol{\theta} \qquad (29)$$

A. Given a magnetic equilibrium, solving equation (26) proceeds as follows:

1. Trajectory integral method: mathematically speaking, this is the method of characteristics applied to the hyperbolic partial differential equation (26). The solution is written as

$$f_s(\mathbf{r},\mathbf{p},t) = -q_s\int_{-\infty}^{t} \{\mathbf{E}(\mathbf{r}',t')+\mathbf{v}(t')\times\mathbf{B}(\mathbf{r}',t')\}.\frac{\partial}{\partial \mathbf{p}'} f_{s0}(\mathbf{r}',\mathbf{p}')\,dt' \qquad (30)$$

i.e. as an integral along the unperturbed particle orbit reaching position **r** with a momentum **p** at time t; **r'** and **p'** are the particle position and momentum at the earlier instant t'. This expression exhibits the nonlocal nature of the interactions between particles and waves in the long mean free path regime: *a priori*, each instant of the past can influence the present. Equation (30) is also manifestly causal. However, the infinitely remote lower limit of integration requires careful justification: one assumes the rf perturbation to have been switched on adiabatically from a vanishing initial value. Enlightening discussions of the effect of nonzero initial conditions can be found in e.g. [6, 7]. The remainder of the derivation is a suitable evaluation of the integral. Here are a few key elements:

- The integrand is highly oscillatory (a tremendous number of wave and cyclotron periods occur during a typical particle transit around the machine);
- Phase memory is lost in some more or less remote past (under the action of collisions and/or nonlinear effects).
- Wave dispersion along the equilibrium magnetic field plays an essential role (associated component of the wavenumber: $k_{//}$), because a particle may travel many wavelengths along $\mathbf{B_0}$ before the phase relationship between wave and cyclotron gyration is lost;
- The hf fields are expanded in Taylor series around the gc. The parameter of the expansion is the ratio (Larmor radius of the particle / characteristic hf lengthscale perpendicularly to $\mathbf{B_0}$). This operation singles out the contributions of the different cyclotron harmonics;
- One finally obtains the perturbed distribution function f_s, which in general emerges as a linear integro-differential operator applied to the hf fields, and also depends linearly on the derivatives of the equilibrium distribution (see equation (30)).

2. Hamiltonan formalism:

Here, the key step is the expansion of f_s and the perturbed Hamiltonian (28) in harmonics of the three angle variables (once more a highly nontrivial operation!). Equation (27) then immediately yields

$$f_{s\boldsymbol{\ell}} = -\boldsymbol{\ell}.\frac{\partial f_{s0}}{\partial \mathbf{J}}\frac{\delta\mathcal{H}_{\boldsymbol{\ell}}}{\omega-\boldsymbol{\ell}.\boldsymbol{\Omega}} \qquad (31)$$

where $\boldsymbol{\ell}$ is a triplet of harmonic indices. The associated resonance condition, generally different from equation (1) in inhomogeneous plasmas, will be discussed in Part III:

$$\omega = \boldsymbol{\ell}.\boldsymbol{\Omega} \qquad (31')$$

B. The plasma dielectric tensor

Given equilibrium distribution functions f_{s0} (most typically Maxwellians), one evaluates the velocity moment (14) of the perturbed distribution function f_s, and obtains the hf current density as a linear functional of **E**, i.e. the plasma hf constitutive relation. This yields the conductivity tensor $\boldsymbol{\sigma}_s$ of each species, generally an integro-differential operator:

$$\mathbf{j}_s(\mathbf{r}) = \boldsymbol{\sigma}_s.\mathbf{E} \qquad (32)$$

In the case of homogeneous plasmas, Fourier analysis in the 3 space dimensions yields

$$\mathbf{j}_s(\omega,\mathbf{k}) = \boldsymbol{\sigma}_s(\omega,\mathbf{k}).\mathbf{E}(\omega,\mathbf{k}) \qquad (33),$$

and the constitutive relation is local in Fourier space, a well-known property of linear systems with constant coefficients. The detailed derivation of $\boldsymbol{\sigma}_s$ in homogeneous Maxwellian plasmas is presented by E. Westerhof [3]. The derivation for inhomogeneous plasmas will be sketched in Part III. If the inhomogeneity is strong, equation (33) doesn't hold, because the various spatial Fourier modes are mutually coupled; if it is sufficiently weak, the homogeneous results can be used locally in the geometric optics approximation. From the conductivities of the various species, one obtains the plasma (relative) dielectric tensor:

$$\boldsymbol{\mathcal{E}} = \mathbf{I} + \frac{i}{\omega\varepsilon_0}\sum_s \boldsymbol{\sigma}_s \qquad (34)$$

C. The wave equation

Using Maxwell's equations for the hf fields, we eliminate **B** and obtain a wave equation for the hf electric field alone:

$$\nabla\times(\nabla\times\mathbf{E}) = \left(\frac{\omega}{c}\right)^2 \boldsymbol{\mathcal{E}}.\mathbf{E} + i\omega\mu_0\mathbf{j_a} \qquad (35)$$

(As for other anisotropic and dispersive dielectrics, the displacement vector is $\mathbf{D} = \varepsilon_0\,\boldsymbol{\mathcal{E}}.\mathbf{E}$.) Note that, once an explicit expression has been obtained for the dielectric tensor, the problem of wave propagation and absorption in the plasma (3 space variables) becomes decoupled from the 3 momentum (or momentum-like) variables. The dielectric tensor fully summarizes the plasma kinetic behaviour.

D. Galerkin formulation of the wave equation

Let us take the scalar product of the wave equation (35) by the complex conjugate of an arbitrary (but sufficiently well-behaved) 'test' vector field **F**, and integrate by parts over a control volume $\mathcal{V}$:

$$\frac{i}{2}\int_{\mathcal{V}}\left\{\frac{1}{\omega\mu_0}(\nabla\times\mathbf{F})^{*}.(\nabla\times\mathbf{E})-\omega\varepsilon_0\,\mathbf{F}^{*}.\mathbf{E}\right\}dr^3+\sum_s\mathcal{W}_{\mathbf{FE}s}=-\frac{1}{2}\int_{\mathcal{V}}\mathbf{F}^{*}.\,\mathbf{j_a}\,dr^3-\frac{1}{2}\int_S\left(\mathbf{E}^{*}\times\mathbf{H}\right).\mathbf{n}\,dr^2 \quad (36)$$

We have introduced the global rf response of the plasma:

$$\mathcal{W}_{\mathbf{FE}s}\equiv\frac{1}{2}\int_{\mathcal{V}}\mathbf{F}^{*}.\mathbf{j}_s\,dr^3=\frac{q_s}{2}\int_{\mathcal{V}}dr^3\int dp^3\;f_s(\mathbf{r},\mathbf{p})\,\mathbf{F}^{*}.\mathbf{v} \quad (37)$$

Requiring equation (36) to hold for arbitrary **F** (within a suitable set of test functions, see [19]) is another valid formulation of Maxwell's equations, particularly useful in finite element numerical applications, and also of high interest for theoretical developments. Poynting's theorem (the time-averaged hf power balance) is obtained for **F**=**E**.

IV. QUASILINEAR FOKKER-PLANCK EQUATION

This section only gives a glimpse at the slow timescale t_0 of equation (23). The interested reader will consult e.g. [6, 14] and references therein. Here the kinetic equation (10) is averaged over all the rapid timescales of the system, an operation noted << >> (this implies averaging over some spatial coordinates as well). In an axisymmetric tokamak, the result is known as the bounce-averaged quasilinear Fokker-Planck equation, and describes the evolution of the equilibrium under the action of wave heating and collisional relaxation; it has the following form:

$$\frac{\partial f_{s0}}{\partial t}=\left\langle\left\langle-q_s(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\frac{\partial f_s}{\partial\mathbf{p}}\right\rangle\right\rangle+\sum_{s'}C(f_{s0},f_{s'0})+S \quad (39)$$

The averaged term is the quasilinear diffusion operator, quadratic in the hf fields and a linear second order operator in f_{s0}. Contributions thereto are once again associated with a resonance condition, eq.(1) or eq.(31'), respectively for homogeneous and inhomogeneous plasmas.

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