HERMITIAN MOMENT APPROACH TO THE CLASSICAL TRANSPORT PROPERTIES OF A WEAKLY COUPLED MAGNETIZED PLASMA

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ABSTRACT

A complete description of a system in equilibrium is provided by the Grand Canonical Distribution. But, systems are generally not in statistical equilibrium. We shall consider the case of an ideal gaz of charged particles. The linear theory of transport determines the 3×1 matrix of dissipative fluxes $\hat{\mathbf{J}}_r$ namely, the electric current and the electronic and ionic heat fluxes, in terms of a 3×1 matrix of thermodynamic forces $\hat{\mathbf{X}}$ defined by the electric field and the gradient of the densities and temperatures. The components of the 3×3 matrix of tensors $\hat{\mathbf{L}}_{rs}$ of the linear flux-force relations $\hat{\mathbf{J}}_r = \sum_{s=1}^{9} \hat{\mathbf{L}}_{rs} \hat{\mathbf{X}}$ define the set of transport coefficients. They are evaluated for an ion-electron magnetized plasma in the framework of the statistical mechanics of charged particles starting from the Landau kinetic equation.

I. INTRODUCTION: TRANSPORT IN A CLASSICAL GAZ

Each particle of an ideal gaz moves independently. Thus, the total distribution function is the one-particle distribution function. It is therefore enough to establish a non equilibrium distribution for a single particle. Boltzman proposed to find the distribution function from an equation similar to the Liouville equation. The state of a particle is given by its three coordinates (x, y, z) and the three momentum components (p_x, p_y, p_z). We can work in a $6-$Dim space of cartesian coordinates $\mathbf{r}$ and velocity $\mathbf{v}$. The classical distribution function $f(\mathbf{r}, \mathbf{v})$ is then defined as

$$f(\mathbf{r}, \mathbf{v}) = \text{nbr of particles in vol. element } d\mathbf{r}d\mathbf{v}\,. \qquad (1)$$

In absence of collision, the Liouville theorem says that the distribution is conserved in time as the volume element evolves along a flowline:

$$f(\mathbf{r} + d\mathbf{r}, \mathbf{v} + d\mathbf{v}) = f(\mathbf{r}, \mathbf{v})\,. \qquad (2)$$

In presence of collisions:

$$f(\mathbf{r} + d\mathbf{r}, \mathbf{v} + d\mathbf{v}) - f(\mathbf{r}, \mathbf{v}) = dt \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}. \qquad (3)$$

A first order expansion of this equation leads to the well known Boltzman transport equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \mathbf{a} \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \qquad (4)$$

where $\mathbf{a}$ denotes the acceleration $d\mathbf{v}/dt$. The role of the particle collisions is to restore the equilibrium which was perturbed by the external forces. Assuming that f does not vary greatly from its equilibrium value f_0, the collision term can be written (τ-approximation) as:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\frac{(f - f_0)}{\tau}\,, \qquad (5)$$

where τ is the relaxation time depending in general on $\mathbf{r}$ and $\mathbf{v}$. We often have to consider an isothermal system with a gradient in the particle concentration (density). The steady state Boltzman equation in the τ-approximation then reads

$$v_x \frac{df}{dx} = -\frac{(f - f_0)}{\tau}\,. \qquad (6)$$

The non-equilibrium distribution function f varies in the x-direction. To first order, with $\partial f/\partial x$ replaced by $\partial f_0/\partial x$, we find

$$f_1 \approx f_0 - \tau\, v_x \frac{df_0}{dx}\,. \qquad (7)$$

In the classical limit

$$f_0 = \exp -(\mathcal{E} - \mu)/kT\,, \qquad (8)$$

where μ is the chemical potential. Then,

$$f_1 \approx f_0 - \tau\, v_x \frac{df_0}{d\mu} \frac{d\mu}{dx}\,. \qquad (9)$$

The density of flux of particles j_x in the x-direction is

$$j_x = \int v_x\, f\, dv_x = -\frac{d\mu}{dx} \int \frac{\tau\, v_x^2}{kT}\, f_0\, dv_x\,. \qquad (10)$$

In general $\tau = A\, v^s$ with $s = 0, -1, -2, ...$ When τ is independent of v_x ($s = 0$), we have

$$j_x = -\frac{d\mu}{dx} \frac{2\tau}{mkT} \int \frac{mv_x^2}{2}\, f_0\, dv_x = -\frac{d\mu}{dx} \frac{2\tau}{mkT} \frac{nkT}{2} \quad (11)$$

where $n = \int f_0 dv_x$. Since $\mu = kT\, \ln n + cst$, the isothermal diffusion (flux) appears to be induced by the density gradient (force) through the well known a Fick's law:

$$j_x = -\frac{d\mu}{dx} \frac{n\tau}{m} = -D\, \frac{dn}{dx}\,, \qquad (12)$$

where the diffusivity D (transport coefficient) is :

$$D = \tau\, \frac{kT}{m}\,. \qquad (13)$$

When $s = -1$, $\tau \approx 1/v = l/v$ where l is the mean free path, a Fick's law is again obtained with a diffusivity $D = \frac{1}{3}\, l\, \bar{c}$. We now turn to a more systematic method for determining the transport coefficients. This method should be able to deal correctly with coulomb collisions between the charged particles constituting the plasma and with strong but uniform magnetic field.

II. STATISTICAL DESCRIPTION OF A PLASMA

We restrict ourselves to the study of a *quiescent fully ionized plasma* of electrons of charge $e_e = -e$ and of positively charged ions which can be nuclei of hydrogen or of one of its isotope, deuterium or tritium of charge $e_i = +Ze$. Here e is the *absolute* value of the charge of the electron. The electron-to-ion mass ratio is approximately $1/1836 = 5.45 \times 10^{-4}$. We thus assume that $\mu = m_e/m_i << 1$ and neglect all quantities that are of order μ. The total number of particles of species α will be denoted by N_α. In a fully ionized plasma, *i.e.* in absence of ionization and recombination processes, these numbers are constant in time for both species. Assuming the plasma is *globally neutral* the total negative charge of the electrons exactly compensates the total positive charge of the ions, $N_e = Z\, N_i$. The total number of particles is $N = \sum_\alpha N_\alpha$.

Each macroscopic quantity $B^\alpha(\mathbf{x}, t)$ is an average of a corresponding microscopic quantity $b^\alpha(\mathbf{q}, \mathbf{v}; \mathbf{x})$, the process of averaging being defined by

$$B^\alpha(\mathbf{x}, t) \;=\; \int d\mathbf{q}\, d\mathbf{v}\, b^\alpha(\mathbf{q}, \mathbf{v}; \mathbf{x}) f^\alpha(\mathbf{q}, \mathbf{v}, t)\,, \quad (14)$$

where f^α is the reduced one-particle distribution function of the particles of species α. Here $b^\alpha(\mathbf{q}, \mathbf{v}; \mathbf{x})$ can be any function of the phase space.

The plasma is assumed to be weakly coupled (a situation which arises when the number of particles N_D in a sphere of radius equal to the Debye length λ_D is much larger than one: $N_D >> 1$; this condition can be met either for very low densities or for very high temperatures) and that a kinetic regime exists (the two-body correlation function $g^{\alpha\beta}$ reduces to a functional of the reduced one-particle distribution function). The reduced one-particle distribution function then satisfies a kinetic equation:

$$\partial_t f^\alpha(\mathbf{v}, \mathbf{x}, t) \;=\; \Phi^\alpha + \mathcal{F}^\alpha + \mathcal{K}^\alpha\,, \qquad (15)$$

where $\Phi^\alpha = -\mathbf{v} \cdot \nabla f^\alpha$ is the free flow term, $\mathcal{F}^\alpha$ is the Lorentz force produced by the self-consistent and by the external electro-magnetic fields. The last term $\mathcal{K}^\alpha$ is the collisional contribution which is assumed to be of the *Landau* form [1]:

$$\mathcal{K}^{\alpha\beta} \;=\; 2\pi e_\alpha^2 e_\beta^2 \ln\Lambda \int d\mathbf{v}_2\, m_\alpha^{-1} \frac{\partial}{\partial v_{1r}} G_{rs}(\mathbf{\Delta v}) \times \left(m_\alpha^{-1} \frac{\partial}{\partial v_{1s}} - m_\beta^{-1} \frac{\partial}{\partial v_{2s}} \right) f^\alpha(1) f^\beta(2)\,, \quad (16)$$

where $\mathbf{\Delta v} = \mathbf{v}_1 - \mathbf{v}_2$ and the *Landau tensor* $G_{rs}(\mathbf{a})$ is given by

$$G_{rs}(\mathbf{a}) = \frac{a^2 \delta_{rs} - a_r a_s}{a^3}. \qquad (17)$$

The *Coulomb logarithm* $\ln\Lambda$ depends on both the electron and the ion temperatures:

$$\ln\Lambda = \ln \frac{\frac{3}{2}(T_e + T_i)\lambda_D}{Ze^2}. \qquad (18)$$

The characteristic time of the like-particle collision terms or *relaxation times* are defined as usual by

$$\frac{1}{\tau_e} \;=\; \frac{4\sqrt{2\pi}}{3} \frac{Z^2 e^4 n_i \ln\Lambda}{m_e^{1/2} T_e^{3/2}}\,, \qquad (19)$$

$$\tau_i \;=\; \mu^{-1/2} Z^{-2} \big(\frac{T_i}{T_e}\big)^{3/2} \tau_e\,. \qquad (20)$$

The collision operator (16) has some important properties. First, as expected from the kinetic theory, it is a functional of the reduced one-particle distribution function. Second, it conserves the number of particles of each species, the total momentum and the total energy (the 5 collisional invariant). Thus

$$\int d\mathbf{v}_1\, \mathcal{K}^\alpha \;=\; 0, \qquad \alpha = e, i \qquad (21)$$

$$\sum_\alpha m_\alpha \int d\mathbf{v}_1\, v_{1r}\, \mathcal{K}^\alpha \;=\; 0, \qquad r = 1, 2, 3 \quad (22)$$

$$\sum_\alpha \frac{1}{2} m_\alpha \int d\mathbf{v}_1\, v_1^2\, \mathcal{K}^\alpha \;=\; 0. \qquad (23)$$

Recalling the collision operator is a sum of two terms, it is easy to show that like-particle collisions separately conserve the number, the momentum and energy of the particles of species α. If $\psi = 1$, v_r, or v^2, we have

$$\int d\mathbf{v}\,\psi\,\mathcal{K}^{\alpha\alpha} \;=\; 0. \qquad (24)$$

Unlike-particle collisions satisfy the following properties

$$\begin{aligned} \int d\mathbf{v}_1\,\mathcal{K}^{ei} &= \int d\mathbf{v}_1\mathcal{K}^{ie}\,, \\ \int d\mathbf{v}_1\, m_e v_{1r}\mathcal{K}^{ei} &= -\int d\mathbf{v}_1\, m_i v_{1r}\mathcal{K}^{ie}\,, \\ \int d\mathbf{v}_1\, m_e v_1^2\mathcal{K}^{ei} &= -\int d\mathbf{v}_1\, m_i v_1^2\mathcal{K}^{ie}\,. \end{aligned} \qquad (25)$$

The unlike-particle collision terms can be rewritten as

$$K^{\alpha\beta} = \frac{2\pi Z^2 e^4 \ln\Lambda}{m_e^2}\partial_{1r}\Phi_r^{\alpha\beta}\,, \qquad (26)$$

with

$$\begin{aligned} \Phi_r^{ei} &= \int d\mathbf{v}_2\, G_{rs}(\boldsymbol{\Delta}\mathbf{v})(\partial_{1s} - \mu\partial_{2s}) f^e(\mathbf{v}_1) f^i(\mathbf{v}_2)\,, \\ \Phi_r^{ie} &= \mu\int d\mathbf{v}_2\, G_{rs}(\boldsymbol{\Delta}\mathbf{v})(\mu\partial_{1s} - \partial_{2s}) f^i(\mathbf{v}_1) f^e(\mathbf{v}_2)\,, \end{aligned}$$

where $\partial_{1s} = \partial/\partial v_{1s}$. Because Φ_r^{ie} is proportional to μ which is very small, the ion-electron collision terms will be neglected. We now assume that the thermal velocity of the electrons $\mathbf{u}^e$ is much larger than the thermal velocity of the ions $\mathbf{u}^i$. We also assume that the difference between the ion and electron mean velocity is much smaller than the electron thermal velocity:

$$|\mathbf{u}^{ei}| \equiv |\mathbf{u}^e - \mathbf{u}^i| << \big(3\frac{T_e}{m_e}\big)^{1/2}\,. \qquad (27)$$

In such conditions the unlike-particle collision terms can be "expanded" starting from a zero-order Lorentz gaz approximation of the plasma where the electrons collide with heavy stationary ions. For this reason, the expansion procedure first proposed by Braginskii [2] is named Lorentz process in [3]. The Landau tensor is first written as

$$G_{rs}(\mathbf{v}_1 - \mathbf{v}_2) = G_{rs}[(\mathbf{v}_1 - \mathbf{u}^e) - (\mathbf{v}_2 - \mathbf{u}^e)]\,. \qquad (28)$$

We now note that on average $(\mathbf{v}_2 - \mathbf{u}^e) << (\mathbf{v}_1 - \mathbf{u}^e)$. The Landau tensor G_{rs} can now be expanded in a Taylor series:

$$\begin{aligned} G_{rs}(\mathbf{v}_1 - \mathbf{v}_2) &= G_{rs}(\mathbf{v}_1 - \mathbf{u}^e) \qquad (29)\\ &- (v_{2n} - u_n^e)\,\partial_{1n} G_{rs}(\mathbf{v}_1 - \mathbf{u}^e) + ... \end{aligned}$$

With this expansion the electron-ion collision operator (26) becomes

$$\begin{aligned} \Phi_r^{ei} &= n_i\,[G_{rs}(1) + u_n^{ei}\,\partial_{1n}G_{rs}(1) + ...]\partial_{1s} f^e(\mathbf{v}_1) \\ &- \mu n_i\,[\partial_{1s}G_{rs}(1) + ...] f^e(\mathbf{v}_1)\,. \end{aligned} \qquad (30)$$

It is now much simpler to show that the electron and ion temperatures are quasi-collisional invariant quantities *i.e.* the contributions of the collisions in their equations of evolution can be neglected (in the case of the electron-temperature it is the ion-electron collisions which are negligible, whereas in the case of the ion-temperature the collisional contribution is non-linear in the moments and thus beyond the scope of the linear theory developed here). The ion momentum, contrarily to the electron momentum, is not a collisional invariant quantity. As a result, the electric current is not a collisional invariant.

III. ORDERING OF THE CLASSICAL THEORY OF TRANSPORT

The characteristic times (inversed here) we have to consider are

$$\begin{aligned} \tau_{T_\alpha}^{-1} &\equiv \big(\frac{T_\alpha}{m_\alpha}\big)^{1/2}\frac{1}{T_\alpha}|\nabla T_\alpha|, \qquad \tau_\rho^{-1} \equiv \big(\frac{T_e}{m_e}\big)^{1/2}\frac{1}{\rho}|\nabla\rho|, \\ \tau_u^{-1} &\equiv |\nabla\mathbf{u}|, \qquad \tau_E^{-1} \equiv \big(\frac{m_e}{T_e}\big)^{1/2}\frac{e}{m_e}\mathbf{E}. \end{aligned} \qquad (31)$$

By definition, the hydrodynamical time τ_H will be the smallest of all these times. The classical theory of transport assumes that the ratio of the collision time to the hydrodynamic time is very small: $\tau_\alpha/\tau_H << 1$. The plasma is thus assumed to evolve on two well separated time scales, a fast one on the collision time scale and a slow one on the hydrodynamical time scale.

IV. DERIVATION OF THE TRANSPORT EQUATIONS

The most important moments (polynomials in the components of the velocity) of the distribution function (rdf) are the local *number density*, the *flux of particles* and the *total kinetic energy density*. They are respectively given by

$$\begin{aligned} n_\alpha(\mathbf{x},t) &\equiv \int d\mathbf{v} f^\alpha(\mathbf{x},\mathbf{v};t)\,, && (32)\\ \Gamma^\alpha(\mathbf{x},t) &\equiv n_\alpha\mathbf{u}^\alpha(\mathbf{x},t) = \int d\mathbf{v}\mathbf{v} f^\alpha(\mathbf{x},\mathbf{v};t)\,, && (33)\\ n_\alpha\mathcal{E}^\alpha(\mathbf{x},t) &\equiv \frac{1}{2}\int d\mathbf{v} v^2 f^\alpha(\mathbf{x},\mathbf{v};t)\,. && (34) \end{aligned}$$

The moments introduced here define the *two-fluids description of the plasma.*

A. The local plasma equilibrium state

We now consider the separation of the dynamics in slow and fast processes: like-particle collision processes are fast processes compared to any of the other processes: unlike-particle collisions and hydrodynamical processes. The electron-electron and the ion-ion collisions bring the plasma in a short time to a state of *local plasma equilibrium*, defined by the equations

$$\mathcal{K}^{ee} = 0, \qquad \mathcal{K}^{ii} = 0\,. \tag{35}$$

The local distribution functions solutions of these equations are

$$f^{\alpha 0}(\mathbf{v},\mathbf{x},t) = n_\alpha(\mathbf{x},t)\left(\frac{m_\alpha}{T_\alpha(\mathbf{x},t)}\right)^{3/2}\phi^{\alpha 0}(\mathbf{c},\mathbf{x},t)\,, \tag{36}$$

where

$$\mathbf{c} = \big(\frac{m_\alpha}{T_\alpha}\big)^{1/2}[\mathbf{v} - \mathbf{u}^\alpha(\mathbf{x},t)]\,, \tag{37}$$

and $\phi^{\alpha 0}$ is a Gaussian

$$\phi^{\alpha 0}(\mathbf{c},\mathbf{x},t) = (2\pi)^{-3/2}\exp(-c^2)\,. \tag{38}$$

B. The moment representation

The state of the plasma, after a short transition time, remains close to the local plasma equilibrium. For this reason, the local plasma equilibrium is called a *reference state.* The distribution functions can then conveniently be written in the form

$$f^\alpha(\mathbf{v},\mathbf{x},t) = f^{\alpha 0}(\mathbf{v},\mathbf{x},t)\big[1 + \chi^\alpha(\mathbf{v},\mathbf{x},t)\big]\,, \tag{39}$$

where $f^{\alpha 0}$ is the distribution function (36) of the reference state. The functions $\chi^\alpha(\mathbf{v},\mathbf{x},t)$ represent the deviation of the local distribution functions f^α from the distributions of the reference state. We have seen that the local plasma equilibrium distribution function depends on five parameters n_α, $\mathbf{u}^\alpha$ and T_α. If the plasma is in a state of local equilibrium, these parameters coincide precisely with the two-fluid variables. In an arbitrary state this coincidence not necessarily holds. It is however possible to construct a representation (39) in which the five parameters are the exact values of the density, velocity and temperature. This implies

$$\int d\mathbf{v} f^\alpha = \int d\mathbf{v} f^{\alpha 0} = n_\alpha$$

$$\int d\mathbf{v} v_r f^\alpha = \int d\mathbf{v} v_r f^{\alpha 0} = n_\alpha u_r^\alpha \tag{40}$$

$$\frac{1}{3}m_\alpha \int d\mathbf{v}|\mathbf{v} - \mathbf{u}^\alpha|^2 f^\alpha = \frac{1}{3}m_\alpha \int d\mathbf{v}|\mathbf{v} - \mathbf{u}^\alpha|^2 f^{\alpha 0} = n_\alpha T_\alpha \tag{41}$$

These conditions are constraints on the deviations χ^α: $\int d\mathbf{v} f^{\alpha 0}\chi^\alpha = 0$, $\int d\mathbf{v} f^{\alpha 0} v_r \chi^\alpha = 0$ and $\int d\mathbf{v} f^{\alpha 0} v^2 \chi^\alpha = 0$. They represent a characteristic feature of the Chapman-Enskog method [4]. The main problem is now to find an approximate solution for the unknown functions χ^α. The basic idea is to expand the functions χ^α in a series of orthogonal polynomials. The kinetic equation then provides a series of equations for the coefficients of the expansion [5]. Recently, a 3-dimensional expansion in irreducible tensorial Hermite polynomials was proposed by Balescu [3].

The unknown functions are expanded in *irreducible tensorial Hermite polynomials* denoted by $H^{(m)}_{r_1\ldots r_q}(\mathbf{c})$:

$$\begin{aligned}\chi^\alpha(\mathbf{c},\mathbf{x},t) &= \sum_{n=0}^{\infty} h^{\alpha(2n)}(\mathbf{x},t)\,H^{(2n)}(\mathbf{c}) \\ &+ \sum_{n=0}^{\infty} h_r^{\alpha(2n+1)}(\mathbf{x},t)\,H_r^{(2n+1)}(\mathbf{c}) \\ &+ \sum_{n=1}^{\infty} h_{rs}^{\alpha(2n)}(\mathbf{x},t)\,H_{rs}^{(2n)}(\mathbf{c})\,.\end{aligned} \tag{42}$$

The subscripts are vectorial or tensorial indices whereas the superscripts give the order of the polynomials. The coefficients $h^{\cdots}_{\cdots}$ are called the Hermitian moments of the distribution function. They are classified in scalar Hermitian moments $h^{\alpha(2n)}$ (corresponding to the polynomials $H^{(0)} = 1$, $H^{(2)} = \frac{1}{\sqrt{6}}(c^2 - 3)$, ...), vectorial Hermitian moments $h_r^{\alpha(2n+1)}$ (corresponding to the vectorial polynomials $H_r^{(1)} = c_r$, $H_r^{(3)} = \frac{1}{\sqrt{10}}c_r(c^2 - 5)$, ...), traceless tensor Hermitian moments $h_{rs}^{\alpha(2n)}$ (corresponding to the traceless second rank polynomials $H_{rs}^{(2)} = \frac{1}{\sqrt{2}}(c_r c_s - \frac{1}{3}c^2)$, ...), and so on. The moments are related to the original function by

$$h^{\alpha(m)}_{r_1\ldots r_q}(\mathbf{x},t) = \int d\mathbf{c}\, H^{(m)}_{r_1\ldots r_q}(\mathbf{c})\,\phi^0(\mathbf{c})\chi^\alpha(\mathbf{c},\mathbf{x},t)\,. \tag{43}$$

The equations of evolution of the moments $h^{\cdots}_{\cdots}$ of the distribution function are obtained by taking the moments of the kinetic equation. They thus result from an integration over $\mathbf{c}$ of the product of the kinetic equation (15) with the corresponding polynomial $H^{\cdots}_{\cdots}$. The moment method also makes use of the constraints (24). In the Hermitian moment method, the constraints simply require that three Hermitian moments are identically zero for every acceptable deviation χ^α :

$$h^{\alpha(0)} = 0, \qquad h_r^{\alpha(1)} = 0, \qquad h^{\alpha(2)} = 0. \tag{44}$$

Two moments $h_{rs}^{\alpha(2)}$ and $h_r^{\alpha(3)}$ have a very important physical meaning, the first one is related to the *pressure tensor* by $\pi_{rs}^\alpha = \sqrt{2}\, n_\alpha T_\alpha h_{rs}^{\alpha(2)}$ whereas the second one is related to the *heat flux* by $q_r^\alpha = \sqrt{\frac{5}{2}}\, m_\alpha \big(\frac{T_\alpha}{m_\alpha}\big)^{3/2} n_\alpha h_r^{\alpha(3)}$.

The accuracy of the derivation of the transport coefficients depends on the truncature of the polynomial expansion of the distribution function. In the *thirteen moments approximation* the moments included in the expansion are $h_r^{\alpha(1)}$,$h_r^{\alpha(3)}$, $h^{\alpha(0)}$, $h^{\alpha(2)}$ and $h_{rs}^{\alpha(2)}$. Thus:

$$f^\alpha(\mathbf{v},\mathbf{x},t) = f^{\alpha 0}(\mathbf{v},\mathbf{x},t)\big[1+\mathbf{h}^{\alpha(3)}(\mathbf{x},t)\cdot\mathbf{H}^{(3)}(\mathbf{v}) + \mathbf{h}^{\alpha(2)}(\mathbf{x},t) : \mathbf{H}^{(2)}(\mathbf{v})\big]\,. \quad (45)$$

The *twenty-one moment approximation* also retains the next higher order vectorial moment $h_r^{\alpha(5)}$ and traceless moment $h_{rs}^{\alpha(4)}$. A final remark concerns the lowest order vectorial quantities $h^{\alpha(1)}$ which are absent in the moment method. One of them is formally reintroduced through a notation for the current density $j_r = en_e\big(\frac{T_e}{m_e}\big)^{1/2} h_r^{(1)}$. We recall this quantity is not a collisional invariant and thus evolves, like the Hermitian moments, on the fast time scale.

C. Equations of evolution of the moments

The equations of evolution of the moments are obtained by taking the moments of the kinetic equation. These equations are the result of an integration over the velocities of the product of the kinetic equation with all the polynomials retained in a given level of truncature. The results in the 13-moments approximation are listed below

$$\partial_t h_r^{(1)} = \frac{1}{\tau_e} g_r^{(1)} - \frac{e}{m_e c}\epsilon_{rmn} h_m^{(1)} B_n + Q_r^{(1)} + U_r^{(1)} + C_r^{(1)} + N_r^{(1)}\,, \quad (46)$$

$$\partial_t h_r^{\alpha(3)} = \frac{1}{\tau_\alpha} g_r^{\alpha(3)} + \frac{e}{m_e c}\epsilon_{rmn} h_m^{\alpha(3)} B_n + Q_r^{\alpha(3)} + U_r^{\alpha(3)} + D_r^{\alpha(3)} + C_r^{\alpha(3)} + N_r^{\alpha(3)}\,, \quad (47)$$

$$\partial_t h_{rs}^{\alpha(2)} = \frac{1}{\tau_\alpha} g_{rs}^{\alpha(2)} + \frac{2e_\alpha}{m_\alpha c}\mathcal{T}_{rs|pq}\epsilon_{rmn} h_{qm}^{\alpha(2)} B_n + Q_{rs}^{\alpha(2)} + U_{rs}^{\alpha(3)} + D_{rs}^{\alpha(3)} + C_{rs}^{\alpha(3)} + N_{rs}^{\alpha(3)}\,, \quad (48)$$

where the (thermodynamic) source terms $g_{\because}^{\because}$ are given by

$$g_r^{(1)} = \tau_e\big(\frac{T_e}{m_e}\big)^{1/2}\big(\frac{1}{n_e T_e}\nabla_r(n_e T_e) + \mathcal{F}_r\big)\,,$$

$$g_r^{\alpha(3)} = -\tau_\alpha\sqrt{\frac{5}{2}}\big(\frac{T_\alpha}{m_\alpha}\big)^{1/2}\frac{1}{T_\alpha}\nabla_r T_\alpha\,,$$

$$g_{rs}^{\alpha(2)} = -\tau_\alpha\sqrt{2}\mathcal{T}_{rs|pq}\nabla_p u_q\,. \quad (49)$$

Here the Lorentz force is denoted $\mathcal{F}_r = eE_r + \frac{e}{c}\epsilon_{rmn}u_m B_n$ and the *symmetrization operator* by $\mathcal{T}_{rs|pq} = \frac{1}{2}(\delta_{rp}\delta_{sq} + \delta_{rq}\delta_{sp} - \frac{1}{3}\delta_{rs}\delta_{pq})$. Let us consider the first of these moment equations. The different terms (similar in the other moment equations) are given by

$$U_r^{(1)} = \sqrt{2}\big(\frac{T_e}{m_e}\big)^{1/2}\frac{1}{n_e T_e}\nabla_s[n_e(T_e h_{rs}^{e(2)} - \mu T_i h_{rs}^{i(2)})] \quad (50)$$

$$C_r^{(1)} = -\mathbf{u}\cdot\nabla h_r^{(1)} - h_m^{(1)}\nabla_m u_r + \frac{1}{3}h_r^{(1)}\nabla\cdot\mathbf{u} + \frac{\sqrt{2}}{3}h_r^{(1)}h_{mn}^{e(2)}\nabla_m[u_n - \big(\frac{T_e}{m_e}\big)^{1/2}h_n^{(1)}] \quad (51)$$

$$N_r^{(1)} = \big(\frac{T_e}{m_e}\big)^{1/2}\big[\frac{\sqrt{10}}{6}h_r^{(1)}\frac{1}{n_e T_e^{3/2}}\nabla(n_e T_e^{3/2}h_m^{e(3)}) + h_m^{(1)}\nabla_m h_r^{(1)} - \frac{1}{3}h_r^{(1)}T_e^{-1/2}\nabla_m(T_e^{1/2}h_m^{(1)})\big] - \frac{1}{3}h_r^{(1)}h_n^{(1)}Q_n^{(1)} \quad (52)$$

Recalling the main assumption of the classical transport theory, all the "Up" (containing higher order polynomial) and "Down" (containing lower order polynomial), the convective and the non-linear contributions ($U_r^{\because}$, $D_r^{\because}$, $C_r^{\because}$ and $N_r^{\because}$ respectively) are negligible compared to the source terms, the collision terms (or friction terms) $Q_n^{\alpha(n)}$ and the magnetic terms. We recall the friction terms are made up of two contributions corresponding to like-particle and unlike-particle collision processes. If ψ represents any moment in the 13-moments approximation, the frictions are given by

$$Q_\psi^{ee} = -\frac{3\sqrt{2\pi}}{4}\frac{n_e}{Z^2 n_i}\frac{1}{\tau_e}\mathcal{Q}_\psi^{ee}\,, \quad (53)$$

$$Q_\psi^{ii} = -\frac{3\sqrt{2\pi}}{4}\frac{1}{\tau_i}\mathcal{Q}_\psi^{ii}\,, \quad (54)$$

where the dimensionless functions are

$$\mathcal{Q}_\psi^{\alpha\alpha} = \int d\mathbf{c}_1 d\mathbf{c}_2\big(\frac{\partial\psi(\mathbf{c}_1)}{\partial c_{1m}}\big)G_{mn}(\gamma)\times \times\left(\frac{\partial}{\partial c_{1n}} - \frac{\partial}{\partial c_{2n}}\right)\phi^e(\mathbf{c}_1)\phi^e(\mathbf{c}_2) \quad (55)$$

The *electron-ion generalized frictions* are obtained from the *Lorentz form* of the collision operator (30):

$$Q_\psi^{ei} = -\frac{3\sqrt{2\pi}}{4}\frac{1}{\tau_e}\mathcal{Q}_\psi^{ei}\,. \quad (56)$$

Finally we recall that the *ion-electron generalized frictions*, of order μ, are neglected. The linear parts of the frictions can be evaluated explicitly and reduce to

$$\tau_e Q_r^{(1)} = -c_{11}^e h_r^{(1)} - c_{13}^e h_r^{e(3)}, \quad (57)$$

$$\tau_e Q_r^{e(3)} = -c_{31}^e h_r^{(1)} - c_{33}^e h_r^{e(3)}, \quad (58)$$

$$\tau_i Q_r^{i(x)} = -c_{33}^i h_r^{i(3)}, \quad (59)$$

$$\tau_e Q_{rs}^{\alpha(2)} = -c_{22}^\alpha h_{rs}^{\alpha(2)}. \quad (60)$$

For later reference, we also add the generalized frictions related to the particle fluxes:

$$\tau_e Q_r^{e(1)} = c_{11}^e h_r^{(1)} + c_{13}^e h_r^{e(3)}, \tag{61}$$

$$\tau_i Q_r^{i(1)} = -\frac{1}{aA}\big(c_{11}^e h_r^{(1)} + c_{13}^e h_r^{e(3)}\big). \tag{62}$$

where

$$a = \Big(\frac{T_i}{m_i}\frac{m_e}{T_e}\Big)^{1/2}, \qquad A = \frac{1}{Z\mu}\frac{\tau_e}{\tau_i}\,. \tag{63}$$

The numerical factors, which may be depend on the charge number Z, are : $c_{11}^e = 1$, $c_{13}^e = c_{31}^e = 3/\sqrt{10}$, $c_{33}^e = (13 + 4\sqrt{2}Z^{-1})/10$, $c_{33}^i = 0.5657$, $c_{22}^e = 1.2 + 0.8485Z^{-1}$, $c_{22}^i = 0.8485$.

D. Linear transport regime and its solution

We have shown that the linear transport equations are reducible to

$$\begin{aligned}
\partial_t h_r^{(1)} &- \Omega_e \epsilon_{rmn} h_m^{(1)} b_n = Q_r^{e(1)} + \tau_e^{-1} g_r^{(1)}\,,\\
\partial_t h_r^{e(3)} &- \Omega_e \epsilon_{rmn} h_m^{e(3)} b_n = Q_r^{e(1)} + \tau_e^{-1} g_r^{e(3)}\,,\\
\partial_t h_{rs}^{e(2)} &- \Omega_e\big(\epsilon_{rmn} h_{sm}^{e(2)} + \epsilon_{smn} h_{rm}^{e(2)}\big) b_n =\\
&\qquad = Q_{rs}^{e(2)} + \tau_e^{-1} g_{rs}^{e(2)}\,,\\
\partial_t h_r^{i(3)} &- \Omega_e \epsilon_{rmn} h_m^{i(3)} b_n = Q_r^{i(1)} + \tau_i^{-1} g_r^{i(3)}\,,
\end{aligned} \tag{64}$$

where $b_n = B_n/B$ and Ω_α is the Larmor frequency of particles of species α. We have assumed here that the strong magnetic field is of the order of magnitude of the thermodynamic source terms. Defining the three components of a vector $\mathbf{a}$ with respect to the magnetic field $\mathbf{b} = \mathbf{B}/B$ by $\mathbf{a}_\parallel = \mathbf{b}(\mathbf{a}\cdot\mathbf{b}), \mathbf{a}_\perp = \mathbf{b}\times(\mathbf{a}\times\mathbf{b}), \mathbf{a}_\wedge = \mathbf{b}\times\mathbf{a}$ where $\mathbf{a}_\parallel$ is obviously the parallel component of the vector, $\mathbf{a}_\wedge$ is both perpendicular to the magnetic field and tangential to the (local) magnetic surface (thus "perp-tangential") and $\mathbf{a}_\perp$ is perpendicular to the (local) magnetic surface. We get the stationary solution in the form

$$\begin{aligned}
\mathbf{h}^{(1)} &= \big[\tilde\sigma_\parallel \mathbf{g}_\parallel^{(1)} + \tilde\alpha_\parallel \mathbf{g}_\parallel^{e(3)}\big] + \big[\tilde\sigma_\wedge \mathbf{g}_\wedge^{(1)} + \tilde\alpha_\wedge \mathbf{g}_\wedge^{e(3)}\big]\\
&+ \big[\tilde\sigma_\perp \mathbf{g}_\perp^{(1)} + \tilde\alpha_\perp \mathbf{g}_\perp^{e(3)}\big]\,,\\
\mathbf{h}^{e(3)} &= \big[\tilde\alpha_\parallel \mathbf{g}_\parallel^{(1)} + \tilde\kappa_\parallel^e \mathbf{g}_\parallel^{e(3)}\big] + \big[\tilde\alpha_\wedge \mathbf{g}_\wedge^{(1)} + \tilde\kappa_\wedge^e \mathbf{g}_\wedge^{e(3)}\big]\\
&+ \big[\tilde\alpha_\perp \mathbf{g}_\perp^{(1)} + \tilde\kappa_\perp^e \mathbf{g}_\perp^{e(3)}\big]\,,\\
\mathbf{h}^{i(3)} &= \tilde\kappa_\parallel^i \mathbf{g}_\parallel^{i(3)} + \tilde\kappa_\wedge^i \mathbf{g}_\wedge^{i(3)} + \tilde\kappa_\perp^i \mathbf{g}_\perp^{i(3)}\,.
\end{aligned} \tag{65}$$

The tensor moments equations will not be given here (a detailed discussion is however given in [3]). The transport coefficients $\tilde\alpha_\parallel, ...$ are listed below. We use here a matricial notation for the collection of parallel, perp-tangential and perpendicular (radial) components of a vector $\mathbf{y} \equiv \{y_\parallel, y_\wedge, y_\perp\}$:

$$\begin{aligned}
\tilde\sigma &= \Big\{\frac{c_{33}^e}{D_{13}^e}, -\frac{x_e}{G_{13}^e}(c_{33}^{e2} + c_{13}^{e2} + x_e^2), \frac{c_{33}^e D_{13}^e + c_{11}^e x_e^2}{G_{13}^e}\Big\}\,,\\
\tilde\alpha &= \Big\{-\frac{c_{13}^e}{D_{13}^e}, \frac{x_e}{G_{13}^e} c_{13}^e (c_{11}^e + c_{33}^e),\\
&- \frac{c_{13}^e (D_{13}^e - x_e^2)}{G_{13}^e}(c_{33}^{e2} + c_{13}^{e2} + x_e^2)\Big\}\,,\\
\tilde\kappa^e &= \Big\{\frac{c_{11}^e}{D_{13}^e}, -\frac{x_e}{G_{13}^e}(c_{11}^{e2} + c_{13}^{e2} + x_e^2), \frac{c_{11}^e D_{13}^e + c_{33}^e x_e^2}{G_{13}^e}\Big\}\,,\\
\tilde\kappa^i &= \Big\{\frac{1}{c_{11}^i}, -\frac{x_i}{c_{33}^{i2} + x_i^2}, \frac{c_{33}^i}{c_{33}^{i2} + x_i^2}\Big\}\,.
\end{aligned} \tag{66}$$

Recalling the moments are related to the electric current and the electron and ion heat fluxes and the explicit expressions of the thermodynamic sources (49), we see that $\tilde\sigma$ is nothing else than the electrical conductivity. The same type of argument shows that $\tilde\alpha$ and $\tilde\kappa$ are the thermoelectric coefficients and the electron thermal conductivity coefficients respectively. In (66) we have used:

$$x_\alpha \equiv \Omega_\alpha \tau_\alpha\,, \quad D_{pq}^\alpha = c_{pp}^\alpha c_{qq}^\alpha - c_{pq}^{\alpha 2}\,, \tag{67}$$

and a function of the magnetic field defined as

$$G_{13}^e(x_e) = D_{13}^{e2} + (c_{11}^{e2} + c_{33}^{e2} + 2c_{13)}^{e2})x_e^2 + x_e^4 \tag{68}$$

We also note that the dimensional transport coefficients are obtained from the dimensionless ones by

$$\begin{aligned}
\sigma &= \frac{e^2 n_e}{m_e}\tau_e\,\tilde\sigma, \quad \alpha = \sqrt{\frac{1}{5}}\frac{e n_e}{m_e}\tau_e\,\tilde\alpha,\\
\kappa^e &= \frac{5}{2}\frac{n_e T_e}{m_e}\tau_e\,\tilde\kappa^e, \quad \kappa^i = \frac{5}{2}\frac{n_i T_i}{m_i}\tau_i\,\tilde\kappa^i
\end{aligned} \tag{69}$$

E. Short comment on the radial transport

Let us assume that (to first order) the macroscopic quantities like $n_{\alpha 0}$, $T_{\alpha 0}$, $\mathbf{u}_{\alpha 0}$ and Φ_0 (the electric potential) are surface functions depending only on ρ in a toroidal magnetic geometry. They thus satisfy the equations $\mathbf{b}\cdot\nabla n_{\alpha 0} = 0$, $\mathbf{b}\cdot\nabla T_{\alpha 0} = 0$ and $\mathbf{b}\cdot\nabla\Phi_0 = 0$.

The radial components of the vectorial moment equations (65) reduce to

$$\begin{aligned}
h_\rho^{(1)} &= \big[\tilde\sigma_\perp g_\rho^{(1)} + \tilde\alpha_\perp g_\rho^{e(3)}\big]\,,\\
h_\rho^{e(3)} &= \big[\tilde\alpha_\perp g_\rho^{(1)} + \tilde\kappa_\perp^e g_\rho^{e(3)}\big]\,,\\
h_\rho^{i(3)} &= \tilde\kappa_\perp^e g_\rho^{e(3)}\,.
\end{aligned} \tag{70}$$

Thus the radial transport in the classical theory of transport is controlled by the perpendicular transport coefficients. We now note that the transport coefficients depend on the poloidal angle only through the magnetic field. This dependency can be eliminated by a surface averaging. This operation does not act on the thermodynamic source terms which are surface functions and we get

$$\begin{aligned} < h_\rho^{(1)} > &= \left[< \tilde{\sigma}_\perp > g_\rho^{(1)} + < \tilde{\alpha}_\perp > g_\rho^{e(3)} \right], \\ < h_\rho^{e(3)} > &= \left[< \tilde{\alpha}_\perp > g_\rho^{(1)} + < \tilde{\kappa}^e_\perp > g_\rho^{e(3)} \right], \\ < h_\rho^{i(3)} > &= < \tilde{\kappa}^i_\perp > g_\rho^{i(3)}, \end{aligned} \tag{71}$$

where the transport coefficients are now the surface average of the coefficients given in (66). For comparison with the neo-classical transport theory we consider the limit of very strong magnetic field $x_\alpha >> 1$ which are are readily obtained from (66):

$$\begin{aligned} < \tilde{\sigma}_\perp > &= c^e_{11} < x_e^{-2} >, \quad < \tilde{\alpha}_\perp > = c^e_{13} < x_e^{-2} >, \\ < \tilde{\kappa}_\perp > &= c^i_{33} < x_i^{-2} >. \end{aligned} \tag{72}$$

Thus all the perpendicular transport coefficients are of order x_α^{-2} thus inversely proportional to the square of the magnetic field strength (called quasi-linear scaling), a property which will be recovered in the framework of the neoclassical transport theory.

F. Discussion and conclusion

In absence of magnetic field, the unperturbed system is isotropic thus the transport coefficients are scalars: $\sigma_\perp = \sigma_\parallel$ and as expected $\sigma_\wedge = 0$ (the transport in the magnetic surface is produced by the magnetic field) and similar relations for the other coefficients. The transport equations thus reduce to

$$\begin{aligned} \mathbf{j} &= \sigma_\parallel \hat{\mathbf{E}} + \alpha_\parallel(-\nabla T_e), \quad \mathbf{q}^e = \alpha_\parallel T_e \hat{\mathbf{E}} + \kappa^e_\parallel(-\nabla T_e) \\ \mathbf{q}^i &= \kappa^i_\parallel(-\nabla T_i), \qquad \pi^\alpha = \eta^\alpha_\parallel(-\nu), \end{aligned} \tag{73}$$

where $\hat{\mathbf{E}}$ is the modified electric field defined by

$$\hat{\mathbf{E}} = \mathbf{E} + \frac{1}{c}(\mathbf{u} \times \mathbf{B}) + \frac{1}{en_e}\nabla(n_e T_e). \tag{74}$$

This set of equations satisfies the *Curie's principle* which excludes all linear coupling between fluxes and forces of different tensorial nature. The two distinct temperatures appearing here result from the small electron to ion mass ratio $\mu << 1$. In a spatially homogeneous system, the current is given by the usual Ohm's law but the electric field produces a heat flux $\mathbf{q}^e = \alpha_\parallel T_e \hat{\mathbf{E}}$ (the *Peltier effect*). Another simple situation arises when there is no electric field and the electrons are in *mechanical equilibrium.* Then the electron heat flux is given by the usual *Fourier law* and the electron temperature gradient produces an electric current $\mathbf{j} = -\alpha_\parallel \nabla T_e$ (the *thermoelectric effect*).

In presence of a constant magnetic field all the transport coefficients are tensors. The 3×3 matrix of transport coefficients has the Onsager symetric form [6]:

$$\mathbf{L} = \begin{pmatrix} L_\perp(\mathbf{B}) & L_\wedge(\mathbf{B}) & 0 \\ -L_\wedge(\mathbf{B}) & L_\perp(\mathbf{B}) & 0 \\ 0 & 0 & L_\parallel \end{pmatrix} \tag{75}$$

There appears a privileged spatial direction but the system remains isotropic in the plane perpendicular to the magnetic field. As a result, the transport tensors are invariant under any rotation around the axis $\mathbf{B}$. The non-diagonal components of the tensors lead to interesting effects [6]: an electric field in the x-direction induces a current in the y-direction (*Hall effect*) whereas a temperature gradient $\nabla_x T_\alpha$ induces a heat flux $\mathbf{q}^\alpha_y$ (*Righi-Leduc effect*). Two other effects are also included: an electric field E_x produces a heat flux q^e_y (*Nernst effect*) whereas a gradient of temperature $\nabla_x T_e$ produces an electric current j_y (*Ettinghausen effect*). It can be seen from (66) and (67) that the parallel transport coefficients are independent of the magnetic field. The perpendicular coefficients $L_\perp$ are monotonously decreasing functions of B. The perp-tangential coefficients $L_\wedge$ start from zero, reach a maximum value then decrease for large magnetic field (or small collision frequency!).

We also observe that in the parallel direction, the electron and ion heat conductivities scale as $\kappa^i_\parallel/\kappa^e_\parallel = \mu^{1/2}(\tilde{\kappa}^i_\parallel/\tilde{\kappa}^e_\parallel)$ thus heat is mainly transported by the electrons. In the perpendicular direction, the process is reversed as we have $\kappa^i_\perp/\kappa^e_\perp = \mu^{-1/2}(\tilde{\kappa}^i_\perp/\tilde{\kappa}^e_\perp)$ and heat in the perpendicular direction is transported by the ions. The convergence of the transport coefficients with the truncature of the expansion in Hermite polynomials is given in [3]. The theory presented here, which extends the Grad method to a two component (2-temperature) plasma, is a necessary step for understanding the neoclassical transport theory.

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