NEOCLASSICAL TRANSPORT PROPERTIES OF TOKAMAK PLASMAS

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ABSTRACT

The classical transport theory is strictly valid for a plasma in a homogeneous and stationary magnetic field. In the '60, experiments have shown that this theory does not apply as a local theory of transport in Tokamaks. It was shown that global geometric characteristics of the confining elements have a strong influence on the transport. Three regimes of collisionality are characteristic of the neoclassical transport theory: the banana regime (the electronic diffusion coefficient increases starting from zero), the plateau regime (the diffusion coefficient is almost independent of the collisionality) and the Pfirsch-Schlüter regime (the electronic diffusion coefficient again increases with the collisionality).

I. CHARACTERISTIC PARAMETERS

The neoclassical transport theory takes into consideration four basic characteristic lengths:

a) λ_D the *Debye length* measuring the effective range of correlations between the particles;

b) λ_{mfp} the *mean free path*, measuring the average distance travelled by a particle between two successive collisions;

c) L_H the *hydrodynamic length* related to the spatial variations of the macroscopic quantities.

The *collision-dominated plasma regime* is in the range

$$\lambda_D \ll \lambda_{\text{mfp}} \ll L_H\,; \tag{1}$$

d) As we are considering a magnetized plasma, the Larmor radius ρ_L also plays an essential role. Among the various regimes that may be defined, two are considered in the neo-classical theory of transport:

$$\begin{aligned} &1) \quad \lambda_D \ll \rho_L \quad \ll \quad \lambda_{\text{mfp}} << L_H\,, \\ &2) \quad \lambda_D \ll \rho_L \quad \ll \quad L_H < \lambda_{\text{mfp}}\,. \end{aligned} \tag{2}$$

In both cases, we have $\rho_L/\lambda_D \gg 1$ which insures the validity of the Landau collision term and $\epsilon \equiv \rho_L/L_H \ll 1$ which is characteristic of the drift approximation. The neoclassical theory is thus basically a guiding centre transport theory. Case 1) refers to the *short mean free path regime (collisional regime)* whereas case 2) refers to the *long mean free path regime (collisionless regime)*.

We consider specifically the case of a toroidal axisymmetric magnetic field (see [1]):

$$\mathbf{B} = \frac{R_0 \mathcal{B}_P(\rho)}{l_\rho\, l_\zeta}\mathbf{e}_\theta + \frac{\mathcal{S}(\rho)}{l_\rho}\mathbf{e}_\zeta. \tag{3}$$

where $\mathcal{B}_P$ and $\mathcal{S}$ are the effective poloidal and toroidal magnetic fields respectively, R_0 is the distance of the magnetic axis to the symmetry axis of the torus, $\{\rho, \theta, \zeta\}$ are the toroidal coordinates and $\{l_\rho, l_\theta, l_\zeta\}$ are the three scaling factors associated to the toroidal geometry defined by (3). For later use, we define $\mathrm{S} = \mathcal{S}/(R_0\mathcal{B}_P)$. Guiding centre trajectories in this field belong to either one of the two categories: the passing and the banana trajectories (see *e.g.* [2]).

II. THE KINETIC EQUATION

We assume that the drift regime applies to the whole plasma. As starting point, we consider the Landau kinetic equation adapted to the guiding centre motion. The kinetic equation is expressed in the guiding centre coordinates $(\mathbf{q}, \mathbf{v}) \rightarrow (\mathbf{Y}, \mathcal{E}, M, \Phi)$ where $\mathcal{E}$ is the GC total energy, M the magnetic momentum and Φ the gyrophase. We are then allowed to expand the distribution function (and the kinetic equation) in the guiding centre parameter ϵ:

$$f^\alpha = f_0^\alpha + \epsilon f_1^\alpha + \epsilon^2 f_2^\alpha + \ldots \tag{4}$$

Finally, an averaging over the gyrophase removes the fast variations and we get a zero-order GC kinetic equation which determines the reference state of the plasma

$$\left(\frac{\partial}{\partial t} + \mathbf{U}_\alpha \cdot \frac{\partial}{\partial \mathbf{Y}}\right) \bar{f}_0^\alpha(\mathbf{Y}, \mathcal{E}, M; t) = \mathcal{K}_0^\alpha(\bar{f}_0, \bar{f}_0)\,, \tag{5}$$

and a first-order equation known as the drift kinetic equation (DKE):

$$\mathbf{U}_\alpha \cdot \frac{\partial}{\partial \mathbf{Y}} \bar{f}_1^\alpha \quad + \quad \mathbf{V}_D^\alpha \cdot \frac{\partial}{\partial \mathbf{Y}} f_0^\alpha = (\nu^{\alpha e} + \nu^{\alpha i}) \mathcal{L}_\alpha \bar{f}_1^\alpha \\ + \quad U_\alpha \mathcal{N}_\parallel^\alpha f_0^\alpha - U_\alpha e_\alpha E_\parallel^{(A)} \frac{\partial}{\partial \mathcal{E}} f_0^\alpha . \quad (6)$$

where $\nu^{\alpha\beta}$ are the collision frequencies, and where $\mathbf{E}^{(A)} = -c^{-1}\partial_t \mathbf{A}(\mathbf{Y}, t)$, the drift velocity is $\mathbf{V}_D^\alpha = (m_\alpha \Omega_\alpha)^{-1}\, \mathbf{b} \times \big(m_\alpha U_\alpha^2 \mathbf{b} \cdot \nabla \mathbf{b} + M \nabla B + e_\alpha \nabla \Phi\big)$, and the parallel velocity is $\mathbf{U}_\alpha = 2\sigma m_\alpha{}^{-1} \mathbf{b}\big(\mathcal{E} - e_\alpha \Phi(\mathbf{Y}) - MB(\mathbf{Y})\big)^{1/2}$. Here we use the linearized form of the Landau collision term: $\mathcal{L}^\alpha$ is the pitch angle scattering operator in GC variables:

$$\mathcal{L}_\alpha = \frac{m_\alpha}{B} \big(U_\alpha \frac{\partial}{\partial M} U_\alpha M \frac{\partial}{\partial M}\big), \quad (7)$$

and $\mathcal{N}_\parallel^\alpha(\mathcal{E})$, a functional of the distribution functions, are remainder parts of the collision operators.

A. Reference distribution function

We look for a quasi-stationary solution of (5). This equation is multiplied by $|J_\alpha| \ln \bar{f}_0^\alpha$ (J_α is the Jacobian of the GC transformation) and integrated over the guiding centre variables. Then, going back to particle coordinates, the equation becomes

$$\mathbf{B} \cdot \nabla \left(B^{-1} \int d\mathbf{v}\, v_\parallel \, (\ln \bar{f}_0^\alpha - 1) f_0^\alpha \right) \\ = \int d\mathbf{v}\, \mathcal{K}_0^\alpha(\bar{f}_0, \bar{f}_0) \, \ln \bar{f}_0^\alpha . \quad (8)$$

This equation has the typical form of a magnetic differential equation ($\mathbf{B} \cdot \nabla$ in the LHS). The left hand side of (8) can be set to zero by a surface averaging (see appendix I.A) leaving

$$< \int d\mathbf{v} \ln \bar{f}_0^\alpha \mathcal{K}_0^\alpha(\bar{f}_0, \bar{f}_0) >= 0 \,, \quad (9)$$

which by comparison with the H-theorem for the Landau equation leads to the conclusion that $\mathcal{K}_0^\alpha(\bar{f}_0, \bar{f}_0) = 0$ and therefore that $\bar{f}_0$ is a maxwellian:

$$f^\alpha(\mathbf{q}, \mathbf{v}) = \big(\frac{\beta_\alpha}{\pi}\big)^{3/2} n_{\alpha 0}(\mathbf{q}) \, \Phi^\alpha(\mathbf{q}, \mathbf{v}) \,, \quad (10)$$

with

$$\beta_\alpha(\mathbf{q}) = m_\alpha / 2T_{\alpha 0}(\mathbf{q}) \,, \quad (11)$$
$$\Phi^\alpha(\mathbf{q}, \mathbf{v}) = \exp\big(-\beta_\alpha |\mathbf{v} - \mathbf{u}_0^\alpha(\mathbf{q})|^2\big) . \quad (12)$$

This distribution function must be a surface function and must be independent of the gyrophase Φ. The only way to satisfy both constraints is to have $\mathbf{u}_0^\alpha = 0$ and to have $n_{\alpha 0}$, $T_{\alpha 0}$ and Φ_0, functions of ρ only, satisfying the equations $\mathbf{b} \cdot \nabla n_{\alpha 0} = 0$, $\mathbf{b} \cdot \nabla T_{\alpha 0} = 0$ and $\mathbf{b} \cdot \nabla \Phi_0 = 0$. Instead of the kinetic energy $\mathcal{E}$ and the magnetic moment, it might be useful to use the kinetic energy scaled to the thermal energy (denoted x) and the ratio of the magnetic moment to the kinetic energy (denoted λ):

$$x = \frac{\mathcal{E} - e_\alpha \Phi_E}{T_\alpha}, \qquad \lambda = \frac{M}{\mathcal{E} - e_\alpha \Phi_E} \,. \quad (13)$$

The equilibrium distribution and the parallel velocity then take a very simple form

$$f_0^\alpha(x; \rho) = \big(\frac{\beta^\alpha}{\pi}\big)^{3/2} n_\alpha(\rho) e^{-x} \,, \quad (14)$$
$$U_\alpha(x, \lambda; \theta, \rho) = \sigma \beta_\alpha^{-1/2} \sqrt{x} \sqrt{1 - \lambda B(\theta, \rho)} \,, \quad (15)$$

where σ is the sign of U_α. Clearly λ cannot exceed B^{-1}. There exists a λ_c below which the GC parallel velocity never vanishes. This domain corresponds to the passing trajectories. Above λ_c, the parallel velocity vanish at the poloidal angle $|\theta_*|$, the tips of the banana orbit, where $B(\theta_*, \rho) = \lambda^{-1}$.

B. The drift kinetic equation

The neoclassical transport theory requires - in contrast to the classical theory - the solution of the DKE (6). Its solution can be obtained in two regimes: the long *mfp* and the intermediate *mfp* regimes.

B.1 Long and short mean free path regime.

We assume the relaxation time is much longer than the hydrodynamic time $\delta \equiv \lambda_H^{-1} \ll 1$. This small parameter is used to further expand the distribution function (superscripts between parentheses).

$$\bar{f}_1^\alpha = \bar{f}_1^{\alpha(0)} + \delta\, \bar{f}_1^{\alpha(1)} + ... \quad (16)$$

Because $\partial/\partial \mathbf{Y}$ is of order τ_H^{-1}, the two first orders in δ of the DKE are

$$\mathbf{U}_\alpha \cdot \frac{\partial}{\partial \mathbf{Y}} \bar{f}_1^{\alpha(0)} = -\mathbf{V}_D^\alpha \cdot \frac{\partial}{\partial \mathbf{Y}} \bar{f}_0^\alpha \,, \quad (17)$$
$$\mathbf{U}_\alpha \cdot \frac{\partial}{\partial \mathbf{Y}} \bar{f}_1^{\alpha(1)} = (\nu^{\alpha e} + \nu^{\alpha i}) \mathcal{L}_\alpha \bar{f}_1^{\alpha(0)} \\ + \quad U_\alpha \mathcal{M}_\parallel^{\alpha(0)} f_0^\alpha \equiv \sigma^\alpha \,, \quad (18)$$

where the derivative of f_0^α with respect to $\mathcal{E}$ has been performed leaving $\mathcal{M}_\parallel^{\alpha(0)} \equiv \mathcal{N}_\parallel^{\alpha(0)} - T_\alpha E_\parallel^{(A)}/e_\alpha$.

• The zero-order equation is easily integrated. Use (3), we get:

$$\bar{f}_1^{\alpha(0)}(\mathcal{E}, M, \theta, \rho) = -\,\mathrm{S}\, \frac{\partial f_0^\alpha(\mathcal{E}; \rho)}{\partial \rho} \frac{U_\alpha(\mathcal{E}, M, \theta, \rho)}{\Omega_\alpha(\theta, \rho)} \\ + \quad \bar{G}_1^{\alpha(0)}(\mathcal{E}, M, \rho) \,. \quad (19)$$

The unknown functions $\bar{G}_1^{\alpha(0)}$ are determined from the solubility conditions of the first-order DKE.

• The first-order DKE is also a magnetic differential equation. A solution $\bar{f}_1^{\alpha(1)}$ exists whenever the source term σ^α of the equation satisfies $< \sigma^\alpha >= 0$. Because σ^α depends on the parallel velocity whence on θ, the passing and the trapped guiding centre trajectories have to be considered separately (subscripts P and T respectively). The study of the solubility constraint leads to:

$$\bar{G}_{1P}^{\alpha(0)} = 2V_\alpha(\lambda,\rho)\,\sqrt{x}\,\mathcal{G}^\alpha(x,\rho) f_0^\alpha(x,\rho)\,, \qquad (20)$$

and $\bar{G}_{1T}^{\alpha(0)} = 0$ in the NGC variables $\{x,\lambda\}$. The functions $V_\alpha(\lambda,\rho)$ (the averaged parallel velocity) and $\mathcal{G}^\alpha(x,\rho)$ are respectively:

$$V_\alpha = \frac{\sigma}{2}\beta_\alpha^{-1/2}\mathcal{B}_0\int_\lambda^{\lambda_c} d\lambda\, < \sqrt{1-\lambda B(\theta,\rho)} >^{-1}\,, \qquad (21)$$

$$\mathcal{G}^\alpha = -\frac{\mathcal{K}_\alpha}{l_\rho}\frac{\tau_\alpha}{f_0^\alpha}\frac{\partial f_0^\alpha}{\partial\rho} - \frac{m_\alpha\nu_\alpha}{\mathcal{B}_0} < B\mathcal{M}_\alpha >\,, \qquad (22)$$

where $\sigma = \pm 1$ according to the sign of the parallel velocity, $\mathcal{B}_0 =< B^2 >^{1/2}$ is the surface averaged magnetic field, $\Omega_{\alpha 0}(\rho) = e_\alpha \mathcal{B}_0/(m_\alpha c)$ is the surface averaged Larmor frequency, τ_α are the relaxation times and $\mathcal{K}_\alpha = \mathrm{S}\, l_\rho/\Omega_{\alpha 0}\,\tau_\alpha$. The functions $\mathcal{G}_\alpha$ are only known in terms of $\mathcal{M}_\alpha$. The distribution function in NGC variables (keeping only the dominant contributions in the ϵ and δ expansions) now reduces to:

$$\bar{f}^\alpha(x,\lambda) = \frac{\beta_\alpha^{1/2}}{2\pi}\, n_\alpha\, e^{-x}\big[1 + \bar{\chi}_I^{\alpha(0)} + \bar{\chi}_{IIP}^{\alpha(0)}\big]\,, \qquad (23)$$

where the deviation to the maxwellian r is a sum of two terms:

$$\begin{aligned} \bar{\chi}_I^{\alpha(0)}(x,\lambda) &= \frac{\mathrm{S}\,\beta_\alpha^{1/2}}{\Omega_\alpha\,\tau_\alpha} l_\rho U_\alpha(x,\lambda)\times \qquad (24)\\ &\times \left[g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A} + \sqrt{\frac{2}{5}}\big(x-\frac{5}{2}\big)g_\rho^{\alpha(3)}\right] \\ \bar{\chi}_{IIP}^{\alpha(0)}(x,\lambda) &= 2\Theta(\lambda_c-\lambda)\sqrt{x}V_\alpha(\lambda,\rho)\mathcal{G}^\alpha(x,\rho). \qquad (25)\end{aligned}$$

Here the Heaviside function $\Theta(\lambda_c - \lambda)$ is non zero in the passing domain. The deviation from the maxwellian can now be expanded either in irreducible tensorial Hermite polynomials (in the GC coordinates) or in Laguerre-Sonine polynomials (in the NGC coordinates). The two expansions are inter-related since the expansion of the total deviation χ^α defined by

$$\begin{aligned} \hat{\chi}^\alpha(x) &\equiv \frac{1}{8\pi}B\sum_\sigma\int_0^{B^{-1}} d\lambda\int_0^{2\pi} d\phi\,\sigma\chi^\alpha(x,\lambda,\phi;\sigma) \\ &= \sqrt{x}\sum_m a_m^\alpha L_m^{3/2}\,, \qquad (26)\end{aligned}$$

gives $h_\parallel^{\alpha(2n+1)} = \sqrt{3}\,a_n^\alpha$. Using the Laguerre-Sonine expansion, the parallel components of the vectorial moments are easily obtained. With the decomposition $h_\parallel^{\alpha(2n+1)} = h_{\parallel I}^{\alpha(2n+1)} + h_{\parallel IIP}^{\alpha(2n+1)}$, on the one hand, we have

$$h_{\parallel I}^{\alpha(1)} = \frac{\mathcal{B}_0}{B}\mathcal{K}_\alpha(g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A}),\; h_{\parallel I}^{\alpha(3)} = \frac{\mathcal{B}_0}{B}\mathcal{K}_\alpha g_\rho^{\alpha(3)} \qquad (27)$$

which are completely determined since they depend only on the thermodynamic sources and, on the other hand, using the truncated Laguerre-Sonine expansion of $\mathcal{G}^\alpha(x)$:

$$\mathcal{G}^\alpha(x) \;=\; a_{0\,II}^\alpha\, L_0^{3/2}(x) + a_{1\,II}^\alpha\, L_1^{3/2}(x)\,, \qquad (28)$$

we have $(n = 0,1)$:

$$h_{\parallel\,IIP}^{\alpha(2n+1)} = 2\sqrt{\frac{2}{3}}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\frac{B}{\mathcal{B}_0} f_P\, a_{n\,II}^\alpha\,, \qquad (29)$$

where

$$f_P = \frac{3}{4}\mathcal{B}_0^2\int_0^{\lambda_c} d\lambda\,\lambda < \sqrt{1-\lambda B} >^{-1} \qquad (30)$$

($f_P = 1$ in the case of a straight field). Hirshman and Sigmar [3] called f_P the fraction of passing particles.

The neoclassical theory of transport does not requires the complete solution $f_1^{\alpha(1)}$ of the first order DKE. Only the following quantities are needed:

$$< \mathbf{B}\cdot(\nabla\mathbf{h}^{\alpha(2)}) >= \frac{1}{\sqrt{2}} < \int d\mathbf{v}\, v_\parallel^2\,\bar{\xi}_1^{\alpha(1)} >\,, \qquad (31)$$

$$< \mathbf{B}\cdot(\nabla\bar{\mathbf{h}}^{\alpha(4)}) >=< \int d\mathbf{v}\, v_\parallel^2\big(\beta_\alpha v^2 - \frac{5}{2}\big)\,\bar{\xi}_1^{\alpha(1)} >\,, \qquad (32)$$

where $\bar{\xi}_1^{\alpha(1)} = (2\beta_\alpha/n_\alpha)\,\mathbf{B}\cdot\nabla f_1^{\alpha(1)}$.
Remarkably, using the expansion of $\mathcal{G}^\alpha(x)$ given in (28), they reduce to:

$$< \mathbf{B}\cdot(\nabla\cdot\mathbf{h}^{\alpha(2)}) >= \mathcal{Z}'\sqrt{\frac{1}{3}}\big[\mu_{11}^\alpha a_{0\,II}^\alpha + \mu_{13}^\alpha a_{1\,II}^\alpha\big]\,,$$

$$< \mathbf{B}\cdot(\nabla\cdot\bar{\mathbf{h}}^{\alpha(4)}) >= \mathcal{Z}'\sqrt{\frac{5}{3}}\big[\mu_{31}^\alpha a_{0\,II}^\alpha + \mu_{33}^\alpha a_{1\,II}^\alpha\big]\,, \qquad (33)$$

with $\mathcal{Z}' = 2\mathcal{B}_0\phi f_P/\tau_\alpha$. The quantity denoted ϕ is called *the neoclassical factor* and is defined by

$$\phi = (1-f_P)/f_P\,. \qquad (34)$$

It is zero for a straight field and is otherwise positive. It will appear that all the neoclassical transport coefficients are proportional to ϕ (using the standard magnetic field model, $f_P \approx 1 - 1.469\sqrt{\eta}$ and $\phi \approx 1.469\sqrt{\eta}$ where $\eta = r/R_0$). The pseudo-viscosity coefficients are $\mu_{11}^i = 0.5328$, $\mu_{13}^i = -0.3953$, $\mu_{33}^i = 0.5543$, $\mu_{11}^e = 1 + \mu_{11}^i/Z$, $\mu_{13}^e = -0.9487 + \mu_{13}^i/Z$, $\mu_{33}^e = -1.3 - \mu_{33}^i/Z$.

B.2 Intermediate mean free path regime.

In the intermediate mean free path regime, the collisions may balance the free streaming term already in the leading order in (6). In this regime the DKE can also be solved [6] (for a full discussion see [1]) leading to

$$< \mathbf{B} \cdot (\nabla \cdot \mathbf{h}^{\alpha(2)}) >= \mathcal{Z}" \sqrt{\frac{1}{3}} \big[\lambda_{11}^\alpha a_{0\,II}^\alpha + \lambda_{13}^\alpha a_{1\,II}^\alpha\big] \,,$$

$$< \mathbf{B} \cdot (\nabla \cdot \bar{\mathbf{h}}^{\alpha(4)}) >= \mathcal{Z}" \sqrt{\frac{5}{3}} \big[\lambda_{31}^\alpha a_{0\,II}^\alpha + \lambda_{33}^\alpha a_{1\,II}^\alpha\big] \,, \quad (35)$$

where $\mathcal{Z}" = 2\mathcal{B}_0 \phi_{\alpha *} f_P / \tau_\alpha$. The plateau factor is defined by

$$\phi_{\alpha *} = \frac{\sqrt{\pi}}{4\beta_\alpha} \eta^2 \frac{B_\theta}{l_\theta \mathcal{B}_0} \tau_\alpha \,. \quad (36)$$

The *plateau*-pseudo-viscosity coefficients are $\lambda_{11} = 2$, $\lambda_{13} = \lambda_{31} = 0.6325$, $\lambda_{33} = 2.6$. These coefficients are independent of the species.

III. THE VECTOR MOMENT EQUATIONS

The ordering in the case of vectorial moments is as follows (we consider the two particle fluxes and the two heat fluxes)

- neglect terms of order ϵ^2 : ∂_t
- keep source terms and large magnetic terms
- keep collisional friction term in linear approximation (other terms are of order ϵ^2)
- neglect convective and non-linear terms
- keep "Up" and "Down" terms which are of order ϵ (in the classical theory they were discarded because they also are of order λ_H which is no more an expansion parameter in the neoclassical transport theory).

Compared to the classical transport theory the neoclassical theory has new source terms coming from the "Up" and "Down" terms (see part I). It is for the evaluation of these quantities that we need the solution fo the DKE. The simplified moment vector equations are conveniently written in a compact form ($j = 1, 3$):

$$\Omega_\alpha \tau_\alpha \epsilon_{rmn} h_m^{\alpha(j)} b_n + \tau_\alpha Q_r^{\alpha(j)} + \big(g_r^{\alpha(j)} + \bar{g}_r^{\alpha(j)}\big) = 0 \,, \quad (37)$$

where the pseudo-thermodynamic sources (in the moment equations) [or generalized forces from the point of view of the transport equations] $\bar{g}_r^{\alpha(j)}$ are defined by

$$\bar{g}_r^{\alpha(1)} = -\frac{\beta_\alpha^{-1/2} \tau_\alpha}{n_\alpha T_\alpha} \nabla_s (n_\alpha T_\alpha h_{rs}^{\alpha(2)}), \quad (38)$$

$$\bar{g}_r^{\alpha(3)} = -\frac{\beta_\alpha^{-1/2} \tau_\alpha}{T_\alpha^{7/2}} \sqrt{\frac{2}{5}} \nabla_s (T_\alpha^{7/2} h_{rs}^{\alpha(2)}) \,. \quad (39)$$

We also note the following expressions:

$$< l_\zeta \mathbf{e}_\zeta \cdot \bar{\mathbf{g}}^{\alpha(x)} >= 0 \,, \quad (40)$$

$$< \mathbf{B} \cdot \bar{\mathbf{g}}^{\alpha(1)} >= -\frac{\tau_\alpha}{\beta_\alpha^{1/2}} < \mathbf{B} \cdot \Big(\nabla \cdot \mathbf{h}^{\alpha(2)}\Big) > , \quad (41)$$

$$< \mathbf{B} \cdot \bar{\mathbf{g}}^{\alpha(3)} >= -\frac{\tau_\alpha}{(5\beta_\alpha)^{1/2}} < \mathbf{B} \cdot \Big(\nabla \cdot \bar{\mathbf{h}}^{\alpha(4)}\Big) > , \quad (42)$$

where $\bar{\mathbf{h}}^{\alpha(4)} = \sqrt{7}\mathbf{h}^{\alpha(4)} + \sqrt{2}\mathbf{h}^{\alpha(2)}$. The last two relations give a link between the averaged quantities (33) or (35) evaluated respectively in the long *mfp* and *intermediate* regimes by solving the first-order DKE, and the generalized forces.

A. Radial components of the moment equations

The radial components of the moment equations are obtained by multiplying (46) by $l_\zeta \mathbf{e}_\zeta$:

$$\Omega_\alpha \tau_\alpha l_\zeta \mathbf{e}_\zeta \cdot (\mathbf{h}^{\alpha(x)} \times \mathbf{b}) + \tau_\alpha l_\zeta \mathbf{e}_\zeta \cdot \mathbf{Q}^{\alpha(x)} + l_\zeta \big(\delta_{x,1} \mathbf{e}_\zeta \cdot \mathbf{g}^A + \mathbf{e}_\zeta \cdot \bar{\mathbf{g}}^{\alpha(x)}\big) = 0 \,. \quad (43)$$

Here only the contribution $\mathbf{E}^{(A)}$ of the classical thermodynamic source survives because all the other terms only have radial components. Recalling the expression of the magnetic field, the first term becomes

$$\Omega_\alpha \tau_\alpha l_\zeta \mathbf{e}_\zeta \cdot (\mathbf{h}^{\alpha(x)} \times \mathbf{b}) = \frac{e_\alpha}{m_\alpha c} \frac{R_0 \mathcal{B}_P}{l_\rho} \tau_\alpha \, \mathbf{e}_\rho \cdot \mathbf{h}^{\alpha(x)} \,. \quad (44)$$

The surface average of the generalized force term vanishes according to (40). We are therefore left with

$$\frac{e_\alpha}{m_\alpha c} \frac{R_0 \mathcal{B}_P}{l_\rho} \tau_\alpha < h_\rho^{\alpha(x)} > + \tau_\alpha < l_\zeta \mathbf{e}_\zeta \cdot \mathbf{Q}^{\alpha(x)} > + \delta_{x,1} < l_\zeta \mathbf{e}_\zeta \cdot \mathbf{g}^A >= 0 \,. \quad (45)$$

Going back to the physical coordinates, we get

$$< h_\rho^{\alpha(x)} > + < \frac{1}{\Omega_\alpha} \big(\mathrm{S} l_\rho Q_\parallel^{\alpha(x)} + Q_\wedge^{\alpha(x)}\big\rangle + \delta_{x,1} < \frac{1}{\Omega_\alpha \tau_\alpha} \big(\mathrm{S} l_\rho g_\parallel^A + g_\wedge^A\big) >= 0 \,, \quad (46)$$

where, as before, the definition $\mathrm{S} = \mathcal{S}/(R_0 \mathcal{B}_P)$ has been used. This expression shows that the surface averaging of the radial components of a vectorial moment is the sum of several terms which are usually classified as follows:

$$< h_\rho^{\alpha(x)} > = \delta_{x,1} \mathrm{DR} + \mathrm{CL}^{(x)} + \mathrm{NCL}^{(x)} \quad (47)$$

The first term corresponds to the electric drift flux, the second one relates to the classical flux, and the third one is the neoclassical flux. The three contributions are ($x = 1, 3$):

$$\mathrm{DR} = -\sqrt{2}\,\beta_\alpha^{1/2} < \frac{cE_\wedge^{(A)}}{B} >, \tag{48}$$

$$\mathrm{CL}^{(x)} = - < \frac{1}{\Omega_\alpha} Q_\wedge^{\alpha x} >, \tag{49}$$

$$\begin{aligned}\mathrm{NCL}^{(x)} &= -\mathrm{S}l_\rho < \frac{Q_\parallel^{\alpha(x)}}{\Omega_\alpha} + \delta_{x,1}(2\beta_\alpha)^{1/2}\frac{cE_\parallel^{(A)}}{B} > \\ &\equiv -\mathrm{S}l_\rho \frac{m_\alpha c}{e_\alpha} < J > .\end{aligned} \tag{50}$$

The expression for the neoclassical term can be reorganized. First, we rewrite:

$$< J > = < \frac{1}{B^2} B\big(Q_\parallel^{\alpha(x)} + \delta_{x,1}\frac{\Omega_{\alpha 0}}{(2\beta_\alpha)^{-1/2}}\frac{cE_\parallel^{(A)}}{\mathcal{B}_0}\big) > . \tag{51}$$

Second, we use $B^{-2} = ((B^{-2} - \mathcal{B}^{-2}) + \mathcal{B}^{-2})$. We then get the following decomposition of the NCL contribution:

$$\mathrm{NCL}^{(x)} = \mathrm{PS}^{(x)} + \mathrm{B}^{(x)} \quad [+\,\delta_{x,1}\,\mathrm{MDR}\,]. \tag{52}$$

where we have the Pfirsch-Schlüter fluxes, the banana fluxes and the modified drift fluxes defined respectively by

$$\begin{aligned}\mathrm{PS}^{(x)} &= -\mathcal{K}_\alpha \frac{\tau_\alpha}{\mathcal{B}_0}\mathcal{B}_0^2 < \big(\frac{1}{B^2} - \frac{1}{\mathcal{B}_0^2}\big)\mathbf{B}\cdot\mathbf{Q}^{\alpha(x)} >, \\ \mathrm{B}^{(1)} &= -\mathcal{K}_\alpha \frac{\tau_\alpha}{\mathcal{B}_0} < \mathbf{B}\cdot\big(\mathbf{Q}^{\alpha(1)} + \frac{\Omega_{\alpha 0}}{(2\beta_\alpha)^{-1/2}}\frac{c\mathbf{E}^{(A)}}{\mathcal{B}_0}\big) >, \\ \mathrm{B}^{(3)} &= -\mathcal{K}_\alpha \frac{\tau_\alpha}{\mathcal{B}_0} < \mathbf{B}\cdot\mathbf{Q}^{\alpha(3)} >, \\ \mathrm{MDR} &= -\mathcal{K}_\alpha \frac{\tau_\alpha \Omega_{\alpha 0}}{(2\beta_\alpha)^{-1/2}} < \big(\frac{1}{B^2} - \frac{1}{\mathcal{B}_0^2}\big)\mathbf{B}\cdot c\mathbf{E}^{(A)} > .\end{aligned} \tag{53}$$

It is obvious from the expressions given here that the parallel average electric field must also be considered as a thermodynamic source. The thermodynamic flux which is most naturally related to this force is the parallel component of the electric current. The electric current decomposes in only two terms: the classical and the banana contributions

$$< \frac{B}{\mathcal{B}_0} h_\parallel^{(1)} > \;=\; \mathrm{CL} + \mathrm{B} \tag{54}$$

B. The perpendicular moment equations

The vector moment equations are projected on the perp-tangential direction. The resulting equation for the electric current are

$$\begin{aligned}h_\wedge^{(1)} &= \tilde{\sigma}_\perp \left(g_\wedge^{(1)} - \bar{g}_\wedge^{e(1)}\right) + \tilde{\alpha}_\perp \left(g_\wedge^{e(3)} - \bar{g}_\wedge^{e(3)}\right) \\ &- \tilde{\sigma}_\wedge \left(g_\rho^{(1)} - \bar{g}_\rho^{e(1)}\right) + \tilde{\alpha}_\wedge \left(g_\rho^{e(3)} - \bar{g}_\rho^{e(3)}\right)\end{aligned} \tag{55}$$

We have already shown (paper I) that the classical transport coefficients, denoted here for convenience by $\tilde{L}_\wedge$ and $\tilde{L}_\perp$, scale as follows:

$$\tilde{L}_\wedge \approx \Omega_\alpha^{-1} = O(\epsilon), \qquad \tilde{L}_\perp \approx \Omega_\alpha^{-2} = O(\epsilon^2). \tag{56}$$

We also know the radial components of the thermodynamic forces are dominant. Thus

$$g_\rho^{\alpha(P)} = O(\epsilon^0), \qquad g_\wedge^{\alpha(P)} = O(\epsilon). \tag{57}$$

The equation for $h_\wedge^{(1)}$ then reduces to

$$h_\wedge^{(1)} = -\tilde{\sigma}_\wedge\, g_\rho^{(1)} - \tilde{\alpha}_\wedge\, g_\rho^{e(3)} + O(\epsilon^2). \tag{58}$$

This result shows that the perp-tangential components of the fluxes are driven by the radial components of the thermodynamical sources. All the radial components of the fluxes are of order ϵ^2. To leading order in ϵ, we have: $\tilde{\alpha}_\wedge = 0$ and $\tilde{\sigma}_\wedge = -(\Omega_e\tau_e)^{-1}$. The same analysis also applies to the radial components. In conclusion, we have:

$$h_\wedge^{(1)} = \frac{1}{\Omega_e\tau_e}\, g_\rho^{(1)}, \qquad h_\wedge^{\alpha(3)} = \frac{1}{\Omega_\alpha\tau_\alpha}\, g_\rho^{\alpha(3)}. \tag{59}$$

C. Parallel vector moment equations

We now multiply the vector moment equations (46) by $\mathbf{B}$ and perform a surface averaging:

$$\begin{aligned}< \tau_\alpha \mathbf{B}\cdot\mathbf{Q}^{\alpha(x)} > \;&+\; < \mathbf{B}\cdot\mathbf{g}^{\alpha(x)} > \\ &+\; < \mathbf{B}\cdot\bar{\mathbf{g}}^{\alpha(x)} > = 0.\end{aligned} \tag{60}$$

We note once more that only the contribution of the electric field in the classical source terms survives the surface averaging, this term is denoted $\mathbf{g}^{\alpha(1)A}$. Thus

$$< \tau_\alpha \mathbf{B}\cdot\mathbf{Q}^{\alpha(1)} > + < \mathbf{B}\cdot\mathbf{g}^{\alpha(1)A} > + < \mathbf{B}\cdot\bar{\mathbf{g}}^{\alpha(1)} > = 0,$$

$$< \tau_\alpha \mathbf{B}\cdot\mathbf{Q}^{\alpha(3)} > + < \mathbf{B}\cdot\bar{\mathbf{g}}^{\alpha(3)} > = 0. \tag{61}$$

Taking this expressions into account, the banana contributions (53) to the radial vectorial moments are entirely due to the generalized forces ($n = 1$ or 3):

$$\mathrm{B}^{(n)} = -\frac{\mathcal{K}_\alpha}{\mathcal{B}_0} < \mathbf{B}\cdot\bar{\mathbf{g}}^{\alpha(n)} > . \tag{62}$$

We now give the explicit form of the quasi-transport equations. We recall the expressions of the generalized frictions in terms of the thermodynamic moments (see classical transport):

$$\begin{aligned}c_{11}^e h_\parallel^{e(1)} - a c_{11}^e h_\parallel^{i(1)} - c_{13}^e h_\parallel^{e(3)} &= g_\parallel^{e(1)} + \bar{g}_\parallel^{e(1)}, \\ -c_{11}^e h_\parallel^{e(1)} + a c_{11}^e h_\parallel^{i(1)} + c_{13}^e h_\parallel^{e(3)} &= aA\big(g_\parallel^{e(1)} + \bar{g}_\parallel^{e(1)}\big), \\ -c_{13}^e h_\parallel^{e(1)} + a c_{13}^e h_\parallel^{i(1)} + c_{33}^e h_\parallel^{e(3)} &= aA\big(g_\parallel^{e(3)} + \bar{g}_\parallel^{e(3)}\big),\end{aligned}$$

and

$$c^i_{33} h^{i(3)}_\parallel = g^{i(3)}_\parallel + \bar{g}^{i(3)}_\parallel \, .$$

Remarks:

1) The ion moment $h^{i(3)}_\parallel$ is decoupled;

2) The determinant of the electronic equations is identically zero. There is therefore a solubility condition which expresses the mutual dependence of $h^{e(1)}_\parallel$ and $h^{i(1)}_\parallel$:

$$g^{e(1)}_\parallel + aA g^{i(1)}_\parallel + \bar{g}^{e(1)}_\parallel + aA\bar{g}^{i(1)}_\parallel = 0 \tag{63}$$

This condition expresses the vanishing of the total pressure tensor (a condition of mechanical equilibrium). The solution of the moment equations is

$$\begin{aligned} h^{(1)}_\parallel &= \tilde{\sigma}_\parallel \left(g^{(1)}_\parallel - \bar{g}^{e(1)}_\parallel \right) + \tilde{\alpha}_\parallel \left(g^{e(3)}_\parallel - \bar{g}^{e(3)}_\parallel \right) , \\ h^{e(3)}_\parallel &= \tilde{\alpha}_\parallel \left(g^{(1)}_\parallel - \bar{g}^{e(1)}_\parallel \right) + \tilde{\kappa}^e_\parallel \left(g^{e(3)}_\parallel - \bar{g}^{e(3)}_\parallel \right) , \\ h^{i(3)}_\parallel &= \tilde{\kappa}^i_\parallel \left(g^{i(3)}_\parallel - \bar{g}^{i(3)}_\parallel \right) , \end{aligned} \tag{64}$$

where the coefficients $\tilde{\sigma}_\parallel$, $\tilde{\alpha}_\parallel$, $\tilde{\kappa}^e_\parallel$ and $\tilde{\kappa}^i_\parallel$ are the parallel transport coefficients (see classical transport). If the generalized forces are set equal to zero, the ordinary parallel transport equations are recovered.

The parallel transport equations can now be surface averaged:

$$\begin{aligned} < Bh^{(1)}_\parallel > &= \tilde{\sigma}_\parallel \left(< Bg^{(1)A}_\parallel > - < B\bar{g}^{e(1)}_\parallel > \right) \\ &+ \tilde{\alpha}_\parallel < Bg^{e(3)}_\parallel > , \\ < Bh^{e(3)}_\parallel > &= \tilde{\alpha}_\parallel \left(< Bg^{(1)A}_\parallel > - < B\bar{g}^{e(1)}_\parallel > \right) \\ &+ \tilde{\kappa}^e_\parallel < Bg^{e(3)}_\parallel > , \\ < Bh^{i(3)}_\parallel > &= \tilde{\kappa}^i_\parallel < Bg^{i(3)}_\parallel > , \end{aligned} \tag{65}$$

complemented by the solubility constraint

$$< Bg^{e(1)}_\parallel > + aA < B\bar{g}^{i(1)}_\parallel > = 0 \, . \tag{66}$$

We now have the expressions for the two components of the electric current: its classical contribution is $< Bh^{(1)}_\parallel >^{CL} = \tilde{\sigma}_\parallel < Bg^{(1)A}_\parallel >$ and its banana contribution is $< Bh^{(1)}_\parallel >^{B} = -\tilde{\sigma}_\parallel < B\bar{g}^{e(1)}_\parallel > + \tilde{\alpha}_\parallel < B\bar{g}^{e(3)}_\parallel >$. There is no Pfirsch-Schlüter contribution to the electric current.

D. Zero-divergence constraints

The geometrical constraints imposed by the magnetic field imply some relations between the vector moments. It can be shown that in a toroidal magnetic field, the electric current, the particle fluxes and the heat fluxes have zero divergence:

$$\nabla \cdot \mathbf{j} \;=\; O(\epsilon^2), \quad \nabla \cdot \mathbf{q}^\alpha = O(\epsilon^2), \quad \nabla \cdot \mathbf{\Gamma}^\alpha = O(\epsilon^2),$$

Let us consider the first of these equations. To dominant order, we have

$$\nabla \cdot \mathbf{j}_\parallel + \nabla \cdot \mathbf{j}_\wedge \;=\; O \tag{67}$$

but because $\mathbf{b} = \mathbf{B}/B$ and $\nabla \cdot \mathbf{B} = 0$ we also have

$$\mathbf{B} \cdot \nabla \left(\frac{j_\parallel}{B} \right) \;=\; -\nabla \cdot \mathbf{j}_\wedge \tag{68}$$

We recall $\mathbf{j}_\wedge$ is determined by the radial component of the thermodynamic sources (see (59)). The above equation thus reduces to an equation determining the parallel component of the current. The general solution of this magnetic differential equation (here given in dimensionless form) is now easily obtained. We get:

$$\begin{aligned} h^{(1)}_\parallel &= -\frac{\mathcal{B}_0}{B} \mathcal{K}_e \, g^{(1)P}_\rho + \frac{B}{\mathcal{B}_0} \omega_1 \, , \\ h^{\alpha(1)}_\parallel &= -\frac{\mathcal{B}_0}{B} \mathcal{K}_\alpha \left(g^{\alpha(1)}_\rho - g^{\alpha(1)A}_\rho \right) + \frac{B}{\mathcal{B}_0} \omega^\alpha_1 \, , \\ h^{\alpha(3)}_\parallel &= -\frac{\mathcal{B}_0}{B} \mathcal{K}_\alpha \, g^{\alpha(3)}_\rho + \frac{B}{\mathcal{B}_0} \omega^\alpha_3 \, , \end{aligned} \tag{69}$$

where we have introduced three - *yet undetermined* - dimensionless surface quantities ω^α_i. The zero-divergence constraints provides a partial determination of the parallel fluxes of matter and energy in terms of the thermodynamic forces.

The surface averaging of (69) gives

$$\begin{aligned} < \frac{B}{\mathcal{B}_0} h^{\alpha(1)}_\parallel > &= \mathcal{K}_\alpha < \left(g^{\alpha(1)}_\rho - g^{\alpha(1)A}_\rho \right) > + \omega^\alpha_1 \, , \\ < \frac{B}{\mathcal{B}_0} h^{\alpha(3)}_\parallel > &= \mathcal{K}_\alpha < g^{\alpha(3)}_\rho > + \omega^\alpha_3 \, . \end{aligned} \tag{70}$$

The surface averaged parallel components of the vectorial moments where also obtained in (29) and in (65):

• The comparison with (29) gives the coefficients of the distribution function in terms of ω^α_i $(x = 0, 1)$:

$$a^\alpha_{x\, II} = \frac{\sqrt{3}}{2} \beta^{1/2}_\alpha f_P{}^{-1} \, \omega^\alpha_{(2x+1)} \, ; \tag{71}$$

• The comparison with (65) yields a determination of ω^α_i in terms of the generalized forces:

$$\begin{aligned} a\omega^i_1 - \omega^e_1 + \tilde{\sigma}_\parallel &< \frac{B}{\mathcal{B}_0} \bar{g}^{e(1)}_\parallel > - \tilde{\alpha}_\parallel < \frac{B}{\mathcal{B}_0} \bar{g}^{e(3)}_\parallel > \\ &= \mathcal{K}_e < g^{(1)P}_\rho > + \tilde{\sigma}_\parallel < \frac{B}{\mathcal{B}_0} g^{(1)A}_\parallel > \\ \omega^e_3 + \tilde{\alpha}_\parallel &< \frac{B}{\mathcal{B}_0} \bar{g}^{e(1)}_\parallel > - \tilde{\kappa}_\parallel < \frac{B}{\mathcal{B}_0} \bar{g}^{e(3)}_\parallel > \end{aligned}$$

$$= -\mathcal{K}_e < g_\rho^{e(3)} > + \tilde{\alpha}_\parallel < \frac{B}{\mathcal{B}_0} g_\parallel^{(1)A} >$$

$$\omega_3^i - \tilde{\kappa}_\parallel \quad < \quad \frac{B}{\mathcal{B}_0} \bar{g}_\parallel^{i(3)} > = -\mathcal{K}_i < g_\rho^{i(3)} > \quad (72)$$

It is clear from (72) that the solution of the first order DKE is necessary to get the generalized forces in terms of the classical thermodynamic forces as can be seen by combining (33) or (35) with (41), (42) and (71) together with (72).

IV. THE SURFACE AVERAGED RADIAL FLUXES

We shall not discuss the details of the electric drift flux, which is not a dissipative flux, and of the modified electric drift flux since they can usually be neglected in tokamak configurations (they are respectively of order ϵ^3 and ϵ^4).

A. The classical fluxes.

The classical fluxes are very easily derived. The generalized frictions are functions of the perp-tangential vectorial moments which in turn are given in terms of the radial components of the classical source terms:

$$\begin{aligned}
\mathrm{CL}^{(1)} &= -< \frac{1}{\Omega_e \tau_e} (c_{11}^e h_\wedge^{(1)} + c_{13}^e h_\wedge^{e(3)}) > \\
&= < \frac{1}{(\Omega_e \tau_e)^2} > \big(c_{11}^e g_\rho^{(1)P} - c_{13}^e g_\rho^{e(3)} \big) , \\
\mathrm{CL}^{e(3)} &= -< \frac{1}{(\Omega_e \tau_e)^2} > \big(c_{13}^e g_\rho^{(1)P} - c_{33}^e g_\rho^{e(3)} \big) , \\
\mathrm{CL}^{i(3)} &= < \frac{1}{(\Omega_i \tau_i)^2} > c_{33}^i g_\rho^{i(3)} ,
\end{aligned}$$

where we used the fact that the thermodynamic sources are surface functions.

B. The Pfirsch-Schlüter fluxes.

They are non zero whenever $< B^2 >^{-1} \neq < B^{-2} >$. We recall again the expressions of the generalized fluxes:

$$\mathrm{PS}^{e(1)} = -\mathcal{K}_e < \big(\frac{\mathcal{B}_0^2}{B^2} - 1 \big) \frac{B}{\mathcal{B}_0} (c_{11}^e h_\parallel^{(1)} + c_{13}^e h_\parallel^{e(3)}) > \ .$$

The parallel components of the vectorial moments were obtained in terms of the "poloidal fluxes" which, as well as the source terms, are surface functions. Thus :

$$\mathrm{PS}^{e(1)} = \mathcal{K}_e^2 < \big(\frac{\mathcal{B}_0^2}{B^2} - 1 \big) > (c_{11}^e g_\rho^{(1)P} - c_{13}^e g_\rho^{e(3)})$$

$$-\mathcal{K}_e < \big(\frac{\mathcal{B}_0^2}{B^2} - 1 \big) \frac{B^2}{\mathcal{B}_0^2} > (c_{11}^e (a\omega_1^i - \omega_1^e) + c_{13}^e \omega_3^e) \, .$$

The first term does not contribute as we have the property $< \big(\frac{\mathcal{B}_0^2}{B^2} - 1 \big) \frac{B^2}{\mathcal{B}_0^2} >= 0$. We now recognize the transport equations are similar to the classical ones. Each PS transport coefficient is exactly proportional to the corresponding classical transport coefficient. The proportionality constant is a purely geometrical quantity [5]:

$$\mathrm{PS} = \big(\mathcal{F}^2 \, \frac{\mathcal{G} - 1}{\mathcal{G}} \big) \, \mathrm{CL} \qquad (73)$$

with $\mathcal{G} =< \mathcal{B}_0^2 / B^2 >$ and $\mathcal{F} = \mathcal{S} l_\rho / (R_0 \mathcal{B}_P)$. In the case of the standard magnetic field the Pfirsch-Schlüter transport exceeds the classical one by a factor of the order of 10 ($\mathcal{F}^2 = q^2 \eta^{-2}$, $\mathcal{G} - 1 = 2\eta^2$, $\phi = 1.469\sqrt{\eta}$ with $\eta = r/R_0$ and q the safety factor).

C. The Banana fluxes.

One of the most important result obtained so far is the proof of a linear relationship between the generalized stress tensor and the poloidal fluxes. We recall that a comparison of different expressions for the parallel component of the moments gives in turn a set of relations between the generalized stress tensor, the generalized forces, and the classical thermodynamic forces. The banana contribution is the following:

$$\begin{aligned}
B^{e(x)} &= \mathcal{K}_e \phi \, l_{xE}^e(\phi) \, \hat{g}_\parallel^{(1)A} \\
&\quad + \mathcal{K}_e^2 \phi \big[l_{x1}^{ee}(\phi) g_\rho^{(1)} + l_{x3}^{ee}(\phi) g_\rho^{e(3)} + l_{x3}^{ei}(\phi) aA g_\rho^{i(3)} \big] , \\
B^{i(x)} &= \mathcal{K}_e \frac{\phi}{a} \, l_{xE}^e(\phi) \, \hat{g}_\parallel^{(1)A} \\
&\quad + \mathcal{K}_e^2 \frac{\phi}{a} \big[l_{x1}^{ie}(\phi) g_\rho^{(1)} + l_{x3}^{ie}(\phi) g_\rho^{e(3)} + l_{x3}^{ii}(\phi) aA g_\rho^{i(3)} \big] , \\
B^{(1)} &= \frac{1}{\mathcal{B}_0} \phi l_{EE}^e(\phi) \hat{g}_\parallel^{(1)A} \\
&\quad + \frac{\mathcal{K}_e}{\mathcal{B}_0} \frac{\phi}{a} [l_{E1}^e(\phi) g_\rho^{(1)} + l_{E3}^e(\phi) g_\rho^{e(3)} + l_{E3}^i(\phi) aA g_\rho^{i(3)}] ,
\end{aligned}$$

with $(x = 1, 3)$

$$\begin{aligned}
l_{11}^{ee}(\phi) &= D_e^{-1}(\phi)(\mu_{11}^e + \phi \kappa^e \mathcal{D}_{13}^e) \\
l_{13}^{ee}(\phi) &= l_{31}^{ee}(\phi) = D_e^{-1}(\phi)(\mu_{13}^e + \phi \alpha \mathcal{D}_{13}^e) \\
l_{33}^{ee}(\phi) &= D_e^{-1}(\phi)(\mu_{33}^e + \phi \sigma \mathcal{D}_{13}^e) \\
l_{33}^{ii}(\phi) &= D_i^{-1}(\phi) \mathcal{D}_{13}^i \\
l_{x3}^{ei}(\phi) &= l_{3x}^{ie}(\phi) = d_i(\phi) l_{1x}^{ee}(\phi) \\
l_{xE}^e(\phi) &= -l_{Ex}^e(\phi) = \sigma l_{x1}^{ee}(\phi) - \alpha l_{x3}^{ee}(\phi) \\
l_{3E}^i(\phi) &= -l_{E3}^i(\phi) = \sigma l_{31}^{ie}(\phi) - \alpha l_{33}^{ie}(\phi) \\
l_{EE}(\phi) &= -\sigma l_{1E}^e(\phi) + \alpha l_{3E}^e(\phi) \qquad (74)
\end{aligned}$$

where

$$D_e(\phi) \quad = \quad 1 + \phi(\sigma \mu_{11}^e - 2\alpha \mu_{13}^e + \kappa^e \mu_{33}^e) + \phi^2 \Delta_{13}^e \mathcal{D}_{13}^e \, ,$$

$$\begin{aligned}
D_i(\phi) &= \mu_{11}^i + \phi \kappa^i \Delta_{13}^i \mathcal{D}_{13}^i, \qquad d^i(\phi) = \frac{\mu_{13}^i}{D_i(\phi)} \\
\mathcal{D}_{pq}^\alpha &= \mu_{pp}^\alpha \mu_{qq}^\alpha - \mu_{pq}^\alpha \mu_{pq}^\alpha, \qquad \Delta_{13}^e = \sigma \kappa^e - \alpha^2 (75)
\end{aligned}$$

The dimensionless transport coefficients are obtained in terms of the neoclassical factor ϕ. They all start linearly from the origin and tend toward a limiting value for large ϕ. The asymptotic values are easily obtained: $\Delta_{13}^e = (D_{13}^e)^{-1}$ and $D_e(\phi) \to \phi^2\mathcal{D}_{13}^e/D_{13}^e$, $D_i(\phi) \to \phi\mathcal{D}_{13}^i/D_{13}^i$.

D. The Plateau regime.

The transport equations in the plateau regime are obtained by the following substitutions: the neoclassical factor ϕ, a pure geometrical factor as we know, is replaced by a dynamical factor $\phi_{\alpha*} = \frac{1}{4}\sqrt{\pi}\eta^2\nu_{\alpha*}^{-1}$ where $\nu_{\alpha*}^{-1}$ is a dimensionless collision frequency and the coefficients μ_{pq}^α are replaced by the coefficients λ_{pq}^α. The dependence of $\phi_{\alpha*}$ on the collision frequency is very important: it is responsible for the fact that all the diffusion transport coefficients are independent of the collision frequency (thus the name plateau regime).

V. CONCLUSIONS

The neoclassical theory of transport relates four surface averaged fluxes to four surface averaged thermodynamic forces:

$$J_\mu = \sum_{\nu=1}^{4}(\mathcal{L}_{\mu\nu}^{CL} + \mathcal{L}_{\mu\nu}^{PS} + \mathcal{L}_{\mu\nu}^{B}), \tag{76}$$

with

$$\mathcal{L}^{CL} = \begin{pmatrix} L_{11}^{CL} & L_{12}^{CL} & 0 & 0 \\ L_{21}^{CL} & L_{22}^{CL} & 0 & 0 \\ 0 & 0 & L_{33}^{CL} & 0 \\ 0 & 0 & 0 & L_{44}^{CL} \end{pmatrix}, \tag{77}$$

$$\mathcal{L}^{PS} = \begin{pmatrix} L_{11}^{PS} & L_{12}^{PS} & 0 & 0 \\ L_{21}^{PS} & L_{22}^{PS} & 0 & 0 \\ 0 & 0 & L_{33}^{PS} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{78}$$

$$\mathcal{L}^{B} = \begin{pmatrix} L_{11}^{B} & L_{12}^{B} & L_{13}^{B} & L_{14}^{B} \\ L_{21}^{B} & L_{22}^{B} & L_{23}^{B} & L_{24}^{B} \\ L_{31}^{B} & L_{32}^{B} & L_{33}^{B} & L_{34}^{B} \\ L_{41}^{B} & L_{42}^{B} & L_{43}^{B} & L_{44}^{B} \end{pmatrix}. \tag{79}$$

We note that they are many more cross-effects in the banana matrices than in the other classical or Pfirsch-Schlüter matrices. The general conclusion concerning the banana regime is the appearance of a neoclassical factor ϕ which is approximately the ratio of the number of trapped particles to the number of passing particles. For small ϕ all the banana transport coefficients are linear in ϕ. They combine pseudo-viscosity coefficients and classical coefficients.

The diffusive banana coefficients L_{11}^B, L_{22}^B, L_{33}^B are all of the form

$$L_{rr}^{\alpha\alpha} = A_r\frac{n_\alpha\rho_{\alpha 0}^2}{\tau_\alpha}\mathcal{F}^2\ \phi\ l_{rr}^{\alpha\alpha}(\phi), \tag{80}$$

where A is a real number. They are proportional to the collision frequency, also a property of the classical transport coefficients in the limit of large magnetic fields and of the Pfirsch-Schlüter coefficients. The slope is however different, in the banana regime there is a very strong enhancement of the radial diffusion coefficient and of the thermal conductivities. The amplification factor (compared to the classical coefficients) is $\mathcal{F}^2\phi/\mathcal{G} \approx 1.469q^2\eta^{-3/2}$. This is a pure geometrical effect which goes to zero in the case of a straight magnetic field. In the long mean free path regime the diagonal diffusive transport coefficients strongly dominates the classical and Pfirsch-Schlüter transport. It can also be shown that the particle and heat fluxes due to the diagonal transport are always directed outwards. Furthermore, the ionic heat flux dominates the electronic heat flux (as in the classical transport). The non diagonal transport coefficients L_{13}^{ee}, L_{31}^{ee}, L_{13}^{ei}, L_{31}^{ie}, L_{33}^{ei}, L_{33}^{ie} are of the form

$$L_{rs}^{e\beta} = A_{rs}\frac{n_e\rho_{e0}^2}{\tau_e}\frac{T_\beta}{|Z_\beta|T_e}\mathcal{F}^2\ \phi\ l_{rs}^{e\beta}(\phi). \tag{81}$$

The presence of the cross coefficients L_{13}^{ee} implies that a radial electron temperature gradient produces a radial electron flux: the thermodiffusion effect or Soret effect (directed inwards). The reciprocal effect is due to the coefficient L_{31}^{ee} a radial pressure gradient produces a radial electron heat flux (electronic Dufour effect, directed inwards). The remaining coefficients are new effects: radial electron flux produced by a radial ion temperature gradient (inwards), radial electron heat flux produced by a radial ion temperature gradient (outwards), radial ion heat flux produced by a radial pressure gradient (ion Dufour effect, inwards) and radial ion heat flux produced by a radial electron temperature gradient (outwards). The electrical conductivity L_{EE} is inversely proportional to the collision frequency (different behaviour compared to the diffusion coefficients), it has no amplification factor and its banana contribution is, as expected from the shape of the trajectory, negative. Finally, we consider the electrical cross effects which are of the form

$$L_{rE}^{\alpha} = -A_r n_\alpha c\mathcal{F}\ \phi\ l_{rE}^{\alpha}(\phi). \tag{82}$$

They are independent of the collision frequency. The amplification factor, lower than for the diffusion coefficients, is $\mathcal{F}\phi \approx 1.469q\eta^{-1/2}$. These cross-coefficients lead to effects which have no classical or Pfirsch-Schlüter counterparts because of the decoupling of the transport in the parallel direction:

◇ a parallel electric field produces an inward radial electron flux (taking into account the natural ambipolarity, also a radial ion flux). This effect is known as the *Ware-Galeev pinch effect* [7]. It also produces outward electron and ion heat fluxes (*Ettingshausen effect*).

◇ a parallel electric current is produced by a radial ion heat flux, by a radial electron temperature gradient and by a radial ion temperature gradient.

The limit $\phi \to \infty$ shows some peculiarity of the neoclassical transport. Indeed, in this limit $\phi l^e_{1E} \to 1$, $\phi l^e_{3E} \to 0$ and $\phi l_{EE} \to -\sigma$ so that the total electric conductivity produced by the diagonal transport coefficients vanishes. The electric conductivity is then driven only by the radial pressure gradient (the *Bootstrap current* [8]).

We have thus shown that the neoclassical transport theory, which is now, quite well understood, is able to bring to light very interesting effects associated with the geometry of the confining device.

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REFERENCES

1. Balescu, R. 1988 *Transport processes in plasmas: Neoclassical transport*, North Holland.

2. Misguich, J. H. , Weyssow B. & Balescu R. 1992 *DRFC/CAD report* EUR-CEA-FC-1438.

3. Hirshman, S.P. & Sigmar J.D. 1981 *Nucl. Fusion***21**, 1079.

4. Hinton, F. L. & Hazeltine R.D. 1976 *Rev. Mod. Phys.* **48**, 239.

5. Pfirsch, D. & Schlüter A. 1962 *Report MPI/PA/7/62* (Max-Planck-Institut für Physik und Astrophysik, Garching).

6. Galeev, A. A. & Sagdeev R.Z. 1968 *Sov. Phys. JETP* **26**, 233.

7. Ware, A. A. 1970 *Phys. Rev. Lett.* **25**, 15.

8. Bickerton, R. J. , Connor J.W. & Taylor J.B. 1971 *Nature* Phys. Sci. **229**, 110.